Testing of the inverse problem with constant-background in the presence of damping in a single layer

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ABSTRACT A one-dimensional inverse problem in a medium with constant-background properties is studied in the presence of damping. The inversion formula is obtained using Born's approximation and high frequency assumption. This formula is tested for a medium which contains a layer giving rise to a single step-like change in the wavespeed and damping parameters. The effects of background damping and of bandlimited data are also discussed by taking different values of background damping and bandlimiting.

1. Introduction

The inverse problem plays an important role in mapping the interior of Earth, and geological prospecting and managing information about inaccessible parts. One or more signals are introduced near the surface of the Earth in a region of interest and responses from irregularities or inclusions in the interior are recorded. In case of seismic inversion, a homogenous elastic model with constant density and elastic parameters provides an important benchmark. Under the assumption of constant density, Claerbout (1972) presented an approximation method to the inverse problem for velocity inversion. Gerver (1970) demonstrated that the velocity of propagation can be determined from observations at one point. Cohen and Bleistein (1977), Cohen and Hagin (1985) among others used Born's approximation and high frequency assumptions to study the inverse problem in one and higher dimensions. All these models do not consider the damping of the medium. Zaman and Masood (2002) have introduced the damping effects in the model through damped wave equation (Stakgold, 1968).

Our aim in this paper is to consider an analytical example of a single layer to test the model presented by Zaman and Masood (2002) and to see if further investigations in this direction are profitable. The simplest example of a single layer is considered to test the model. We will also discuss the effects of bandlimiting and see how it changes the result in any real world experiment.

2. Result of one-dimensional direct and inverse problems with damping

We consider the one-dimensional inverse problem of determining variations in propagation speed taking into account damping of the medium. Due to the nature of one dimensional models, more powerful mathematical methods are available as compared to the case of higher dimensional inverse problems. One-dimensional inversion theories are of practical interest in situations where data have only one dimension of variability. Our analysis will begin with a mathematical statement of the forward scattering problem. Our goal is to derive a formula that represents the solution to the inverse scattering problem. In the following, two subsections, the direct and inverse solution representations, are discussed. Our method is based on the linearized inversion associated with Born's approximation. We assume wavespeed and damping parameters are well approximated by their respective background values plus a perturbation. We exploit the high frequency character of seismic data. This enables us to use the WKBJ Green's function in deriving our inversion representation.

2.1. Direct problem

The formulation of the forward scattering problem is carried out in the frequency domain for some field $u(x,x_s,\omega)$. Here, x represents the observation position, while x_s represents the location of the source, and ω represents the frequency. The propagation of the field $u(x,x_s,\omega)$ is governed by the scalar Helmholtz equation

$$\pounds' u\,(x, x_s, \omega) = \frac{d^2\,u\,(x, x_s, \omega)}{dx^2} + \left[\frac{\omega^2 + i\omega\gamma(x)}{\upsilon^2(x)}\right] u\,(x, x_s, \omega) = -\delta\,(x - x_s)\,. \quad (1)$$

The wavespeed and damping of the medium are given by $v(x)$ and $\gamma\,(x)$ respectively. The forcing term on the right-hand side represents an impulse located at the position $x = x_s$. The Helmholtz equation is just the temporal Fourier transform of the scalar wave equation

$$\frac{\partial^2 u\,(x,x_s,t)}{\partial x^2} + \frac{\gamma(x)}{v^2(x)}\frac{\partial u\,(x,x_s,t)}{\partial t} - \frac{1}{v^2(x)}\frac{\partial^2 u\,(x,x_s,t)}{\partial t^2} = -\delta(t)\delta(x-x_s).$$

We are confined to methods that involve the introduction of signals that propagate as waves inside the body. These waves scatter from layers present inside the body and are recorded on the surface of the body. Much can be accomplished by assuming that the rules governing the scattering of waves from boundaries in the Earth are the same as the rules governing the scattering of plane waves from plane boundaries between constant velocity acoustic media.

We assume that u satisfies the radiation condition

$$\frac{du}{dx} \mp i\frac{\omega}{v(x)}u \to 0, \qquad as\ x \to \pm\infty. \quad (2)$$

Physically, the radiation condition insures that the energy from the source is outward propagating at infinity and mathematically, this condition insures that the solution to the Helmholtz equation is unique. A recording instrument is located at the position x_g , which for geophysical applications is assumed to be a geophone. If the source and geophone are located at the same point i.e. $x_g = x_s$, then this constitutes a zero-offset experiment. It is assumed that $\upsilon(x)$

and $\gamma(x)$ are known in the range of distances from $-\infty < x < x^+_g$ and unknown in the range $x^+_g < x < \infty$. The position x^+_g is located to the right of x_g .

Suppose that the wavespeed $\upsilon(x)$ and damping profiles $\upsilon(x)$ can be represented as small perturbations from the background profiles $\upsilon_0(x)$ and $\gamma_0(x)$ respectively. While there are many ways of representing such small perturbations, we choose such representation which preserves the form of the Helmholtz equation. To achieve this, we use the expressions of the form

$$\begin{aligned}\frac{1}{\upsilon^2(x)} &= \frac{1}{\upsilon_0^2(x)}\left[1+\alpha(x)\right],\\ \gamma(x) &= \gamma_0(x)+\varepsilon\gamma(x),\end{aligned} \tag{3}$$

where $\alpha(x)$ and ε are assumed to be small. After some lengthy calculations, (see Appendix), the solution to the direct problem can be written as

$$u_S(0,\omega) = -\frac{1}{4}\int_0^\infty \alpha(x)\, e^{2i\omega x/\upsilon_0} e^{-\gamma_0 x/\upsilon_0} dx. \tag{4}$$

This is the Fredholm integral equation of the first kind. The Fourier-like integral operator appearing in Eq. (4) and its inverse are stable analytically and numerically. In deriving this integral equation, the linearization technique based on Born's approximation is used and infinite bandwidth information in ω is used.

2.2. Inverse problem

Our aim is to find the variation in the velocity $\alpha(x)$ from the integral Eq. (4) using the observed data $u_S(0, \omega)$. The variation in velocity $\alpha(x)$ is zero for $x < 0$, meaning that $\alpha(x)$ behaves as a causal function of the time variable, $2x/\upsilon_0$. It is therefore permissible to extend the lower limit of integration in Eq. (4) to $-\infty$. Thus, $\alpha(x)$ is represented in the inverse Fourier transform integral form as

$$\alpha(x) = \frac{-4e^{\gamma_0 x/v_0}}{\pi\upsilon_0}\int_{-\infty}^{\infty} u_S(0,\omega)\, e^{-2i\omega x/\upsilon_0} d\omega. \tag{5}$$

For the observed data $u_S(0, \omega)$ the variation in wavespeed $\alpha(x)$ can be recovered by this result. In the next section, a simple analytical example is presented to see the form of the recovered profile and the effect of damping on it. As a simple check, when $\gamma_0 = 0$, this result reduces to the constant-background inversion formula (Cohen and Bleistein, 1977).

In making the derivations in Eq. (5), we have used the following forward and inverse temporal Fourier transform conventions

$$f(\omega) = \int_0^\infty F(t)\, e^{i\omega t} dt,$$

$$F(t) = \frac{1}{2\pi} \int_\Gamma f(\omega)\, e^{-i\omega t} d\omega.$$

The forward transform implies that no data exist for $t < 0$, that is, the source is initiated at some finite time that we take to be $t = 0$. This type of forward transform is known as the causal Fourier transform. The choice of the integration path in the inverse transform is such that the result is causal in t-domain. The choice of the contour of the integration, Γ, is such that it passes above all singularities of the integrand. In the expressions of the type (5), we simply used the integration limits of $-\infty$, ∞, however, the reader should be aware of the rule just stated to evaluate these integrals, [see Titchmarsh (1948) for Fourier integrals and Spiegel (1953) for complex variable methods].

3. Constant-background, single-layer

For the one-dimensional problem, the exact scattered field data can be generated analytically for a variety of wavespeed and damping profiles. We consider an analytical example to establish the validity of the recovered profile given by Eq. (5) in the following two subsections.

3.1. Direct problem

The simplest profile to consider is a perturbation of size ε. This step-like wavespeed change, located at the position $x = h$, defines the boundary between two constant wavespeed media and is represented mathematically as:

$$a(x) = \varepsilon v(x) = \varepsilon H(x - h) = \begin{cases} 0 \text{ for } & x < h \\ \varepsilon \text{ for } & x > h \end{cases} \tag{6}$$

$$v(x) = \begin{cases} v_0 & \text{for } x < h \\ v_1 = \dfrac{v_0}{\sqrt{1+\varepsilon}} & \text{for } x > h \end{cases} \tag{7}$$

$$\gamma(x) = \begin{cases} \gamma_0 & \text{for } x < h \\ \gamma_1 & \text{for } x > h \end{cases}. \tag{8}$$

The exact solution to the problem (1) for this wavespeed and damping profiles can be written in terms of the incident field u_I, reflected field u_R, and transmitted field u_T as follows

$$u(x,\omega)=\begin{cases} u_I(x,\omega)+u_R(x,\omega), & x<h \\ u_T(x,\omega), & x>h \end{cases}. \tag{9}$$

A suitable choice of solution is given by

$$u=-\frac{\upsilon_0}{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)}\left[e^{i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)x/\upsilon_0}+A_1e^{i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)(2h-x)/\upsilon_0}\right], \qquad x<h$$

$$u=-\frac{\upsilon_0}{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)}\left[A_2e^{i\omega\left[\left(1+\frac{i\gamma_0}{2\omega}\right)h/\upsilon_0+\left(1+\frac{i\gamma_1}{2\omega}\right)(x-h)/\upsilon_1\right]}\right], \qquad x>h. \tag{10}$$

We require that u and its first derivative be continuous at $x = h$. This requirement leads to the system of equations,

$$A_1-A_2=-1 \tag{11}$$

$$\frac{\left(1+\frac{i\gamma_0}{2\omega}\right)A_1}{v_0}-\frac{\left(1+\frac{i\gamma_1}{2\omega}\right)A_2}{v_1}=\left(1+\frac{i\gamma_0}{2\omega}\right). \tag{12}$$

Solving simultaneously for A_1 and A_2 , we obtain the following solution

$$u_1(x,\omega)=-\frac{v_0e^{i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)x/v_0}}{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)}, \tag{13}$$

$$u_R(x,\omega)=-\frac{v_0Re^{-i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)(x-2h)/v_0}}{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)}, \tag{14}$$

$$u_T(x,\omega)=-\frac{v_0Te^{i\omega\left[\left(1+\frac{i\gamma_0}{2\omega}\right)h/v_0+\left(1+\frac{i\gamma_1}{2\omega}\right)(x-h)/v_1\right]}}{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)}, \tag{15}$$

where the ”reflection coefficient” R and the ”transmission coefficient” T are given by

$$R=\frac{\left(1+\frac{i\gamma_0}{2\omega}\right)-\left(1+\frac{i\gamma_1}{2\omega}\right)\sqrt{1+\varepsilon}}{\left(1+\frac{i\gamma_0}{2\omega}\right)+\left(1+\frac{i\gamma_1}{2\omega}\right)\sqrt{1+\varepsilon}}\,, \tag{16}$$

$$T=\frac{2\left(1+\frac{i\gamma_0}{2\omega}\right)\sqrt{1+\varepsilon}}{\left(1+\frac{i\gamma_0}{2\omega}\right)+\left(1+\frac{i\gamma_1}{2\omega}\right)\sqrt{1+\varepsilon}}\,. \tag{17}$$

The scattered wave u_S (0, ω) needed for Eq. (5) is then the expression for the reflected wave u_R (x, ω) in Eq. (14) , evaluated at x = 0. Thus

$$u_s(0,\omega)=-\frac{v_0 R e^{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)h/v_0}}{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)}\,. \tag{18}$$

3.2. Inverse problem

We substitute Eq. (18) in Eq. (5) to get the integral representation of the wavespeed perturbation as follows

$$\alpha(x)=\frac{4e^{-\gamma_0(h-x)/v_0}}{\pi}\int_{-\infty}^{\infty}\tilde{R}\frac{e^{2i\omega(h-x)/v_0}}{2i\omega}d\omega\,, \tag{19}$$

where

$$\tilde{R}=\frac{R}{1+\frac{i\gamma_0}{2\omega}}\,. \tag{20}$$

Now, we examine the Eq. (19) more closely to see how $\alpha(x)$ can be approximately evaluated. Since we are dealing with high frequency, the expression (20) can be written for small ε as

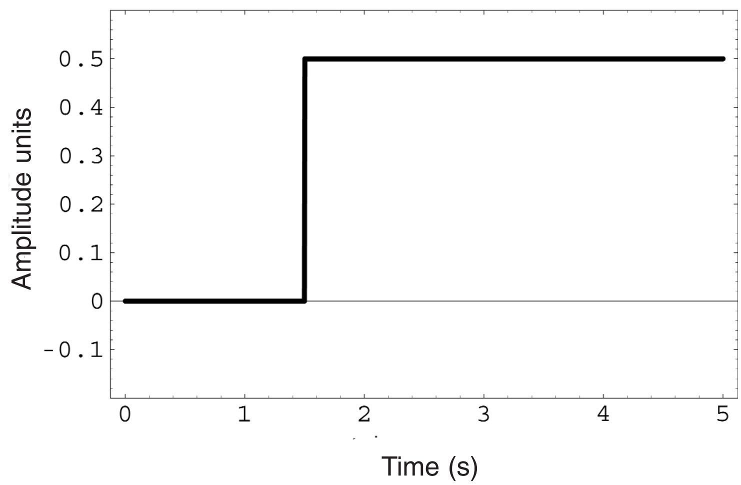

Fig. 1 - A full bandwidth step function without damping with amplitude ε = 0.5 and h = 1.5.

$$\tilde{R} \approx R = \frac{\left(1+\frac{i\gamma_0}{2\omega}\right)-\left(1+\frac{i\gamma_1}{2\omega}\right)\sqrt{1+\varepsilon}}{\left(1+\frac{i\gamma_0}{2\omega}\right)+\left(1+\frac{i\gamma_1}{2\omega}\right)\sqrt{1+\varepsilon}} \approx \frac{1-\sqrt{1+\varepsilon}}{1+\sqrt{1+\varepsilon}} = -\frac{\varepsilon}{4}+O\left(\varepsilon^2\right). \tag{21}$$

The last equality in the above expression follows by using the binomial theorem. As ε is assumed to be small and only leading order terms in ε are retained, the expression (19) with the help of Eq. (21) takes the form

$$\alpha(x) = \frac{\varepsilon e^{-\gamma_0 (h-x)/v_0}}{\pi} \int_{-\infty}^{\infty} \frac{e^{2i\omega(h-x)/v_0}}{2i\omega} d\omega . \tag{22}$$

The expression (22) leads to

$$\alpha(x) = \varepsilon e^{-\gamma_0 (h-x)/v_0} H(x-h). \tag{23}$$

The comparison of the result in Eq. (23) with Eq. (6) shows that the discontinuity is in the right location but the magnitude of the jump is not exact. The jump in Eq. (23) has an extra term $e^{-\gamma_0(x-h)/v_0}$, which was expected due to the damping of the medium under consideration. This result is encouraging at least for this simple example.

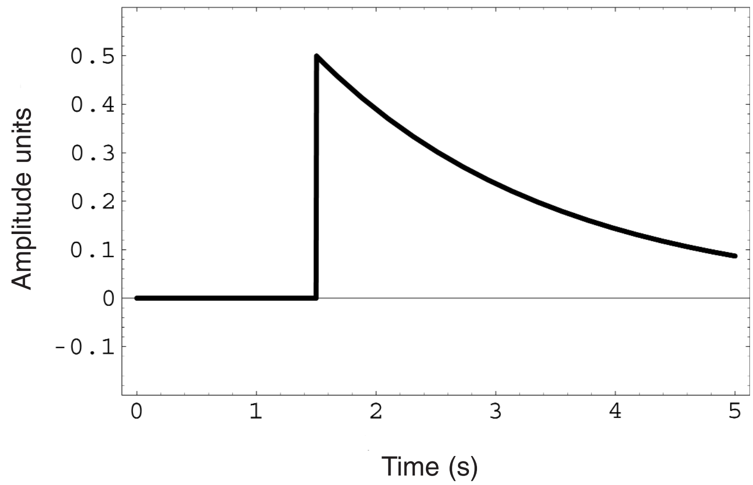

Fig. 2 - A full bandwidth step function (with damping) given by Eq. (23) with $\varepsilon = 0.5$, $\gamma = 0.5$ and $h = 1.5$.

4. Numerical computations

The expression (23) represents the step function with the damping term $e^{-\gamma_0(x-h)/v_0}$ which changes the appearance of the step function. To exhibit this feature, the step function without the damping term is shown in Fig. 1 and the step function with the damping term is shown in Fig. 2. It is clear from these figures that the damping term changes the appearance of the step function.

How would bandlimiting present in any real world experiment change the results? To see this, assume that the observed field is similar to the expression in Eq. (18), but with a multiplicative factor $F(\omega)$, which is a real-valued filter. Because data must be real valued in the space-time domain, the filter, $F(\omega)$, must be symmetric and nonnegative in the ω-domain. The scattered

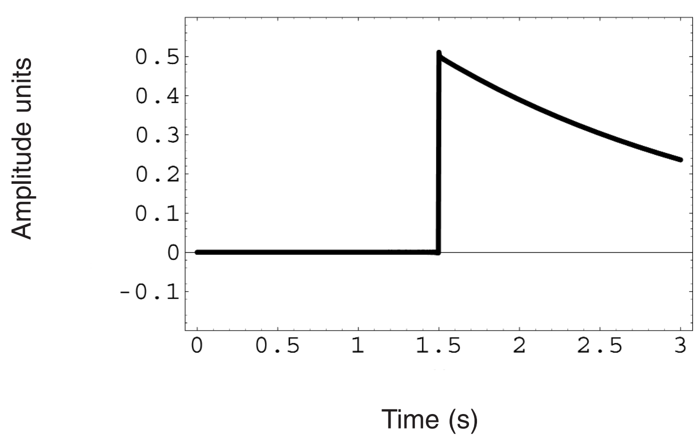

Fig. 3 - A 0-1000 Hz bandwidth representation of the step function with $\varepsilon = 0.5$, $\gamma = 0.5$ and $h = 1.5$.

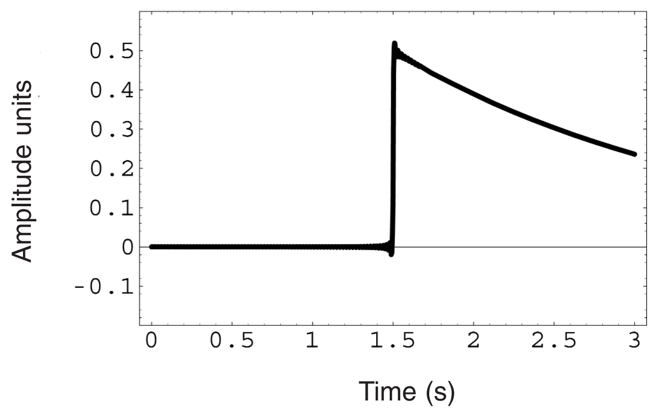

Fig. 4 - A 0-50 Hz bandwidth representation of the step function with $\varepsilon = 0.5$, $\gamma = 0.5$ and $h = 1.5$. 0

field in Eq. (18) can be replaced by

$$u_s(0,\omega) = -F(\omega)\frac{v_0 Re^{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)h/v_0}}{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)}\,. \tag{24}$$

When the bandlimited data given by Eq. (24) is substituted into Eq. (5), the output will be a bandlimited version of $\alpha\,(x)$, represented here by $\alpha_B\,(x)$, given by

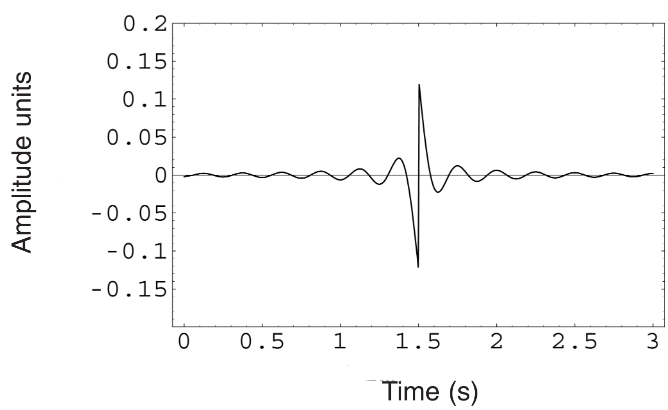

Fig. 5 - A 4-1000 Hz bandwidth representation of the step function with $\varepsilon = 0.5$, $\gamma = 0.5$ and $h = 1.5$.

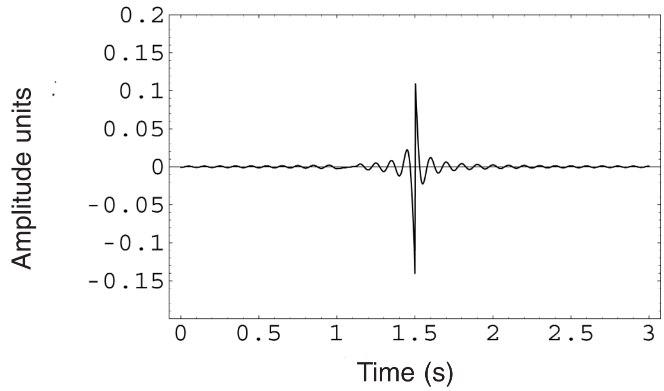

Fig. 6 - A 10-1000 Hz bandwidth representation of the step function with $\varepsilon = 0.5$, $\gamma = 0.5$ and $h = 1.5$.

$$\alpha_B(x) = \frac{\varepsilon e^{-\gamma_0(h-x)/v_0}}{\pi} \int_{-\infty}^{\infty} F(\omega) \frac{e^{2i\omega(h-x)/v_0}}{2i\omega} d\omega \, , \tag{25}$$

We have not tried to find a bandlimited solution directly, instead we introduced the bandlimited data into the solution. We will analyze the effects of bandlimited data on the solution. To find a full-bandwidth solution from bandlimited data is known to be illposed and admits exponentially growing solutions. In Figs. 3, 4, 5 and 6, we show outputs with 0-1000 Hz, 0-50 Hz, 4-1000 Hz and 10-1000 Hz respectively. In Figs. 3 and 4, the result is not seriously degraded. We see in Fig. 5 that only a moderate loss of low frequency makes the step unrecognizable except in the neighborhood of $x = h$, where it exhibits a doublet-like behavior. From Fig. 6, an even further

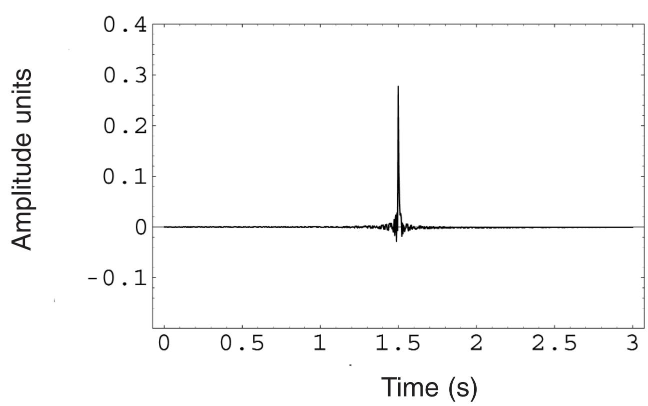

Fig. 7 - A 0-1000 Hz bandwidth representation of the derivative of the step function with $\varepsilon = 0.5$, $\gamma = 0.5$ and $h = 1.5$.

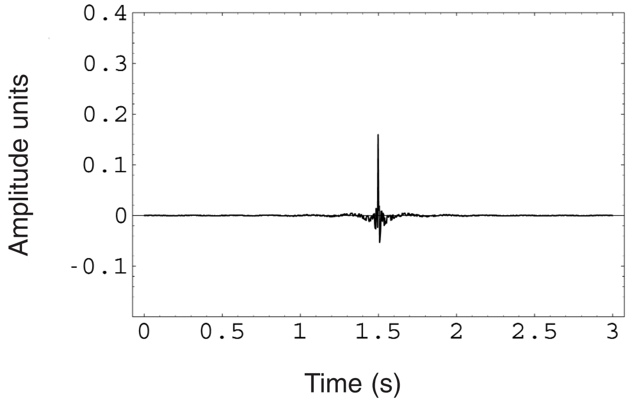

Fig. 8 - A 4-1000 Hz bandwidth representation of the derivative of the step function with $\varepsilon = 0.5$, $\gamma = 0.5$ and $h = 1.5$.

degradation of the output is apparent, although the region of discontinuity is still recognizable. From this analysis, it is clear that slow variations in propagation speed cannot be recovered from high frequency data. From this it is apparent that only the discontinuity of the propagation speed can be recovered for inversion of high frequency data, that is, the reflectors of the unknown medium.

Our primary goal is to seek the reflectors in the unknown medium. If the discontinuities in the medium are of primary interest then there is a better way to process the data. Bandlimited delta functions are easier to identify than are bandlimited step functions. It is not difficult to obtain a bandlimited delta function from Fourier data of bandlimited step functions. This requires introducing a multiplicative factor of $-i\omega$, that can be obtained by the derivative operation. We take the x-derivative of $\alpha_B(x)$ given in (25) to obtain

$$\begin{aligned}\beta_B(x) = & \frac{\varepsilon e^{-\gamma_0(h-x)/v_0}}{\pi v_0}\int_{-\infty}^{\infty} F(\omega) e^{2i\omega(h-x)/v_0} d\omega \,. \\ & -\frac{\varepsilon\gamma_0 e^{-\gamma_0(h-x)/v_0}}{\pi v_0}\int_{-\infty}^{\infty} F(\omega)\frac{e^{2i\omega(h-x)/v_0}}{2i\omega} d\omega \,.\end{aligned} \tag{26}$$

The x-derivative of $\alpha_B(x)$ given above as $\beta_B(x)$ is plotted in Figs. 7 and 8. It is clear from the graphs that the peak reveals the location of the discontinuity. In Fig. 8, the peak is correctly located but its strength is reduced due to the removal of low frequency.

5. Conclusions

The theory was applied to data that would be obtained from piecewise constant wave speed consisting of a single reflector. At least for this simple example, it is demonstrated that the proposed model gives promising results. It is clear from the numerical computations that the damping present in the medium effects the output. The artifacts produced in the figures reveal the location and extent of discontinuities present in the medium. The effects of bandlimiting are also considered and it is shown that the primary goal of recovering the discontinuities of the medium is not affected by considering the bandlimited data.

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REFERENCES

Bender C. M. and Orszag S. A.; 1978: *Advanced mathematical methods for scientists and engineers.* McGraw-Hill, New York, 484 pp.

Claerbout J. F.; 1972: *Downward continuation of moveout corrected seismograms.* Geophysics, **36**,741-768.

Cohen J. K. and Bleistein N.; 1977: *An inverse method for determining small variation in propagation speed.* SIAM J. Appl. Math., **32**(4), 784-799.

Cohen J. K. and Hagin F. G.; 1985: *Velocity inversion using a stratified reference.* Geophysics, **50**, 1683-1700.

Gerver M. L.; 1970: *Inverse problem for the one-dimensional equation.* Geophysics. J. R. Astron. Soc. **21**, 337-357.

Kennet B., N.; 2001: *The seismic wavefield.* Cambridge, New York, 56 pp.

Spiegel M.; 1953: *Complex variables.* Schaum's Outline Series, McGraw-Hill, New York, 172 pp.

Rayleigh L.; 1879: *Contribution.* Philosophical Magazine, **8**, 261.

Stakgold I.; 1968: *Boundary value problems of mathematical physics.* The Macmillan Company, New York, 172 pp.

Titchmarsh E. C.; 1948: *Introduction to the theory of Fourier integrals.* Clarendon, Oxford.

Zaman F. D. and Masood K.; 2002: *An inversion procedure for the recovery of propagation speed and damping of the medium.* Il Nuovo Cimento C, **25**(3), 263-271.

Appendix

In this Appendix details of derivation of the expression (4) are presented. The calculations are carried out for a more general case and then the expression is obtained by appropriate substitutions and assumptions. Introduce variations in damping and sound speed to have the form

$$\gamma(x) = v_0(x) + \varepsilon\gamma(x), \tag{27}$$

$$v(x) = v_0(x) + \varepsilon v(x), \qquad \frac{1}{v^2(x)} = \frac{1}{{v_0}^2(x)}\left[1 - \frac{2\varepsilon v(x)}{v_0(x)}\right]. \tag{28}$$

We have used the same ε for both $\gamma(x)$ and $v(x)$, because if it is ε_1 for $\gamma(x)$ and ε_2 for $v(x)$ then we set $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$. These representations are substituted into Eq. (1) and only linear terms in ε are retained. The resulting equation is

$$\pounds_0 u = \frac{d^2u}{dx^2} + \left[\frac{\omega^2 + i\omega_0(x\gamma)}{\upsilon_0{}^2(x)}\right] u = -\delta\,(x - x_s)$$
$$+\left[-i\varepsilon\omega\gamma(x) + \frac{2\varepsilon\omega^2 v(x)}{\upsilon_0(x)} + \frac{2i\varepsilon\omega\gamma_0(x)v(x)}{\upsilon_0(x)}\right]\frac{u}{\upsilon_0{}^2(x)}\,. \tag{29}$$

Introduce $u_I\,(x, x_S\,,\ \omega)$ as the response to the delta function in the unperturbed medium and $u_S\,(x, x_S\,,\ \omega)$ as everything else, we can write $u = u_I + u_S$ with

$$\pounds_0 u_1(x,x_S,\omega) = -\delta\,(x - x_s)\,, \tag{30}$$

and

$$\pounds_0 u_S(x,x_S,\omega) = \left[-i\varepsilon\omega\gamma(x) + \frac{2\varepsilon\omega^2 v(x)}{\upsilon_0(x)} + \frac{2i\varepsilon\omega\gamma_0(x)v(x)}{\upsilon_0(x)}\right]\frac{u_I(x,x_S,\omega)}{\upsilon_0{}^2(x)}\,. \tag{31}$$

We now construct a Green's function representation of Eq. (31), observed at a point x_g (geophone location). It satisfies the following equation

$$\pounds_0 g(x,x_g,\omega) = -\delta(x - x_g)\,. \tag{32}$$

Note that this Green's function differs from u_I, defined by Eq. (30), only by subscripts s and g. The high frequency approximation allows us to use the asymptotic methods to find high frequency solutions of the forward and inverse problems (Kennet, 2001). A criterion to determine the high frequency is known as Rayleigh Criterion (Rayleigh, 1879), which relates the minimum permissible length scale L to the dominant wavelength Λ by the relation $L \geq \Lambda/4$. This condition can be written in a more suitable form as follows

$$\lambda \geq \pi, \tag{33}$$

where we have used $\Lambda = 2\pi v_0/\omega$, $\omega = 2\pi f$ and introduced a dimensionless parameter $\lambda = 4\pi f L/v_0$. So, in accordance with the Rayleigh Criterion, any dimensionless frequency larger than 3 (or π) is a high frequency.

Since we are concerned only with high frequency inversion, we will use WKBJ approximations (Bender and Orszag, 1978) for u_I and g to the leading order in ω. These leading order approximations are given by

$$u_1(x,x_S,\omega) = -\frac{F(\omega)A(x_S,x)}{2i\omega}\exp\left[i\omega\phi(x_S,x) - \frac{1}{2}\psi(x_S,x)\right], \tag{34}$$

$$g(x,x_g,\omega) = -\frac{A(x_g,x)}{2i\omega}\exp\left[i\omega\phi(x_g,x) - \frac{1}{2}\psi(x_g,x)\right], \tag{35}$$

where for the case of continuous $v_0\,(x)$ and $\gamma_0\,(x)$ the WKBJ amplitude and phase are given by

$$A(x_g,x) = \sqrt{v_0(x_g)v_0(x)}, \qquad \phi(x_g,x) = \int_{x_g}^{x}\frac{dt}{v_0(t)}, \text{ and } \psi(x_g,x) = \int_{x_g}^{x}\frac{\gamma_0(t)dt}{v_0(t)}. \tag{36}$$

The solution of Eq. (31) in terms of Green's function is given by

$$u_S(x_g,x_S,\omega) = -\int_0^{\infty}\left[-i\varepsilon\omega\gamma(x) + \frac{2\varepsilon\omega^2 v(x)}{\upsilon_0(x)} + \frac{2i\varepsilon\omega\gamma_0(x)v(x)}{\upsilon_0(x)}\right] \frac{u_1(x,x_S,\omega)\,g(x,x_g,\omega)}{\upsilon_0^{\,2}(x)}dx. \tag{37}$$

Now using the WKBJ representations (34) and (35) in Eq. (37), and retaining only the leading order terms in ω, we get

$$u_S(x_g,x_S,\omega) = \frac{\varepsilon}{2}\int_0^{\infty}F(\omega)\frac{v(x)}{v_0^3(x)}\,A(x_g,x)A(x_S,x)\exp\left\{i\omega\left[\phi(x_g,x)+\phi(x_S,x)\right]\right\} \exp\left\{\frac{1}{2}\left[\psi(x_g,x)+\psi(x_S,x)\right]\right\}dx\,. \tag{38}$$

This expression is the direct solution of the problem (1). We assume that source and receiver are placed at the same point i.e. $x_S = x_g = 0$, which is the case of a zero offset experiment. If we assume that the background wavespeed $v_0\,(x)$ and background damping $\gamma_0\,(x)$ parameters are constant and use $-2\varepsilon v\,(x)\,/v_0\,(x) = \alpha\,(x)$ the formula (38) reduces to the one given by Eq. (4).

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