

## A theory for wave propagation in porous rocks saturated by two-phase fluids under variable pressure conditions

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**ABSTRACT** In this work we present an extension of Biot's theory to describe wave propagation in elastic and viscoelastic porous solids saturated by two-phase fluids for arbitrarily fixed confining and pore pressure conditions. As the fluids are immiscible, the model takes into account capillary forces. Appropriate bulk and pore volume compressibilities are defined in terms of the coefficients in the stress-strain relations, which lead to a generalization of the classic effective pressure laws for the case of single-phase fluids. Using a Lagrangian formulation, the coupled equations of motion for the solid and the fluid phases are also derived, including dissipative effects due to matrix viscoelasticity and viscous coupling between the solid and fluid phases, which are used to model the levels of wave attenuation and dispersion observed in rocks. Four different body waves can propagate in this type of media, three compressional waves and one shear wave. The sensitivity of the phase velocities and quality factors to variations in saturation and effective pressure in a sample of Boise sandstone saturated by a gas-water mixture is presented and analyzed. Our results suggest that a combined analysis of such measurable quantities can be used as indicators of the saturation and pressure states of a hydrocarbon reservoir.

### 1. Introduction

The analysis of the variation of seismic and mechanical properties of rocks when the saturant fluids are varied in their properties and composition is an active area of research in exploration geophysics, with application in reservoir characterization and engineering and also in petroleum geomechanics.

Theoretical formulations for the study of the deformation and elastic wave propagation in porous rocks with full, partial, multiphase, or segregate fluid saturation have been presented by different authors (see for example Biot 1956b, 1956c, 1962; Toksöz *et al.*, 1976; Dutta and Odé, 1979; Mochizuki, 1982; Berryman *et al.*, 1988). However, none of these models incorporate the capillary forces existing when the pore fluids are immiscible. Consequently, the pressure variations induced by wave propagation in the different fluid phases are considered equal, neglecting possible changes in capillary pressure.

In this work we present a general theory for this kind of problems, which at the same time includes the effects of the ambient overburden pressure and the reference pressures of the immiscible fluids on the mechanical response of the rock. The theoretical basis was given by Santos *et al.* (1990a, 1990b). In addition, generalizing the ideas given in Zimmerman *et al.* (1986)

and in Zimmerman (1991) for single-phase fluids we define six compressibility coefficients to quantify the changes in either the pore volume or the bulk volume associated with changes in confining, wetting fluid and capillary pressures. Using these compressibilities, we established effective pressure laws for this model, showing that the effective pressure in this case depends not only on the wetting fluid pressure but also on the changes in capillary pressure.

For the study of wave propagation processes, two possible sources of energy dissipation are considered in this theory: Biot-type global flow and linear viscoelasticity. The first one is included by means of a viscous dissipation density function in the Lagrangian formulation and involves the relative flow velocities of the two fluids with respect to the solid frame (Santos *et al.*, 1990a). The second one is incorporated by extending the elastic constitutive relations to the linear viscoelastic case by means of a correspondence principle (see Biot, 1956a). In this way the real poroelastic coefficients in the constitutive equations are replaced by complex frequency dependent poroviscoelastic moduli satisfying the same relations as in the elastic case. Viscoelastic behaviour is included in order to model the levels of dispersion and attenuation suffered by the different types of waves when travelling in real rocks. A form of the frequency correction factors for the mass and viscous coupling coefficients in the equations of motion needed in the high-frequency range is also presented. Moreover, generalizing a procedure previously applied to viscoelastic solids by Fabrizio and Morro (1992) and also to porous media saturated by single-phase fluids by Ravazzoli (1995), in the Appendix we analyze the resultant moduli using the laws of continuum equilibrium thermodynamics, leading us to establish some necessary restrictions. Finally, the theory is applied to predict and analyze the influence of the saturation, pore fluid types, capillarity and effective pressures on the phase velocities and quality factors in a sample of Boise sandstone.

## 2. A modified two-phase Biot model

Here we review and extend the model presented by Santos *et al.* (1990a, 1990b) describing the deformation and propagation of waves in an elastic isotropic, homogeneous, porous solid saturated by two immiscible fluids such as oil and water or gas and water. This theory is modified to include explicitly in the formulation the absolute reference pressure of both fluid phases. Consequently, we derive the stress-strain relations and we define a set of elastic moduli and compressibilities for this kind of media.

### 2.1. Derivation of stress-strain relations

When two immiscible fluids occupy the voids of a porous solid one of them (depending on their adhesion tension) tends to preferentially wet the solid surface, spreading over it. In this way we can distinguish a wetting phase and a non-wetting one, which will be denoted with the subscripts (or superscripts) “w” and “n”, respectively. Let  $S_n$  and  $S_w$  denote the averaged wetting and non-wetting fluid saturations, respectively. Further, let us assume that the two fluid phases completely saturate the porous part of the bulk material so that  $S_n + S_w = 1$ .

Let  $u^s$ ,  $\tilde{u}^n$ , and  $\tilde{u}^w$  denote the averaged absolute displacement vectors for the solid, non-wetting and wetting phases, respectively. Let  $\phi$  denote the effective porosity and for  $l = n, w$  set  $u^l = \phi (\tilde{u}^l - u^s)$  and  $\xi^l = -\nabla \cdot u^l$ .

At a reference initial state, we consider a volume  $\bar{V}_b$  of homogeneous bulk material containing fluid volumes  $\bar{V}_n$  and  $\bar{V}_w$  at pressures  $\bar{P}_n$  and  $\bar{P}_w$ . In such a state we have  $\bar{V}_p = \bar{V}_n + \bar{V}_w$ ,  $\bar{S}_l = \bar{V}_l / \bar{V}_p$ ,  $l = n, w$ ,  $\bar{\phi} = \bar{V}_p / \bar{V}_b$ .

Note that for uniform porosity  $\bar{S}_n \xi^n$  and  $\bar{S}_w \xi^w$  represent the change in the corresponding fluid contents per unit volume of bulk material (Santos *et al.*, 1990a). Thus, if  $\Delta V_l^c$  denotes the part of the total change in volume  $\Delta V_l = V_l - \bar{V}_l$  due to changes  $\Delta P_l = P_l - \bar{P}_l$  in the corresponding fluid pressures, since in equilibrium  $\nabla S_l = 0$ , we see that

$$\bar{S}_l \xi^l = (\Delta V_l - \Delta V_l^c) / \bar{V}_b = \bar{\phi} (\Delta V_l - \Delta V_l^c) / \bar{V}_p, \quad l = n, w. \tag{2.1}$$

Let  $K_n = C_n^{-1}$  and  $K_w = C_w^{-1}$  denote the bulk moduli of the non-wetting and wetting fluids, respectively,  $C_n$  and  $C_w$  being the corresponding compressibilities. Then, by definition

$$\Delta V_l^c / \bar{V}_l = -\Delta P_l / K_l, \quad l = n, w. \tag{2.2}$$

Also, neglecting second order terms,  $\Delta V_l = \bar{V}_p \Delta S_l + \bar{S}_l \Delta V_p$ ,  $l = n, w$ , and consequently

$$\xi^l = \bar{\phi} (\Delta S_l / \bar{S}_l + \Delta V_p / \bar{V}_p - \Delta V_l^c / \bar{V}_l), \quad l = n, w, \tag{2.3}$$

where  $\Delta V_p$  denotes the change in pore volume. Setting  $\xi^* = \bar{S}_n \xi^n + \bar{S}_w \xi^w$ , it follows from Eqs. (2.1) and (2.3) that

$$\xi^* = \bar{\phi} (\Delta V_p - \Delta V_n^c - \Delta V_w^c) / \bar{V}_p, \tag{2.4}$$

Next, let  $\tau_{ij} = \bar{\tau}_{ij} + \Delta \tau_{ij}$ ,  $i, j = 1, 2, 3$ , be the total stress tensor of the bulk material,  $\Delta \tau_{ij}$  being the change in the total stress with respect to a reference value  $\bar{\tau}_{ij}$  corresponding to the initial equilibrium state. In the same way,  $P_n = \bar{P}_n + \Delta P_n$  and  $P_w = \bar{P}_w + \Delta P_w$  denote the absolute pressures of the fluid phases,  $\Delta P_n$  and  $\Delta P_w$  representing small increments in the corresponding pressures with respect to their reference values  $\bar{P}_n$  and  $\bar{P}_w$ , and set  $\Delta S_n = S_n - \bar{S}_n = -\Delta S_w$ . Recall that  $P_n$  and  $P_w$  are related through the capillary relation (see Bear, 1972; Scheidegger, 1974; Peaceman, 1977), which can be regarded as an equation of state for this system:

$$P_{ca}(S_n) = P_n - P_w = P_{ca}(\bar{S}_n) + \Delta P_{ca} \geq 0. \tag{2.5}$$

Then, ignoring terms of the second order in  $\Delta S_n$ ,

$$\Delta P_{ca}(\bar{S}_n + \Delta S_n) \cong P_{ca}'(\bar{S}_n) \Delta S_n. \tag{2.6}$$

The function  $P_{ca}(S_n)$  is a positive and strictly increasing function of the variable  $S_n$ . Since in the present analysis we are not considering an imbibition-drainage cycle, it is not necessary to include hysteresis effects in the capillary pressure curve.

The symbols  $S_{rw}$  and  $S_{rn}$  will denote the residual wetting and non-wetting fluid saturations, respectively. The saturation  $S_{rn}$  is the amount of nonwetting fluid that remains in the pores when

the capillary pressure tends to zero and  $S_{rw}$  (also called irreducible saturation) is the remaining wetting fluid when the capillary pressure reaches its maximum value.

We assume that both fluids are allowed to move inside the pore space, so that  $S_{rn} < S_n < 1 - S_{rw}$  (see Collins, 1961; Bear, 1972; Scheidegger, 1974; Peaceman, 1977).

Let us consider a volume  $\Omega$  of bulk material of boundary  $\partial\Omega$  in static equilibrium state under the action of the surface forces  $f_i^s, f_i^n, f_i^w, i = 1, 2, 3$ , acting on the solid and fluid parts of  $\partial\Omega$  per unit area of bulk material. These forces can be written in the form (Santos *et al.*, 1990a)

$$f_i^s = [\tau_{ij} + \bar{\phi} (S_n P_n + S_w P_w) \delta_{ij}] v_j \quad f_i^l = \bar{\phi} S_l P_l \delta_{ij} v_j, \quad l = n, w, \quad (2.7)$$

where  $\delta_{ij}$  denotes the Kronecker symbol and  $v_j$  is the unit normal to  $\partial\Omega$ .

If  $W^* = W^*(\Delta\tau_{ij}, \Delta P_n, \Delta P_w)$  represents the complementary strain energy density of the system, then its complementary potential energy  $V^*$  is given by (see Fung, 1965; Santos *et al.*, 1990a)

$$V^* = \int_{\Omega} W^* dx - \int_{\partial\Omega} (f_i^s u_i^s + f_i^n \tilde{u}_i^n + f_i^w \tilde{u}_i^w) d\sigma, \quad (2.8)$$

where  $dx$  denotes an infinitesimal volume and  $d\sigma$  a surface element in the boundary  $\partial\Omega$ . Here and in what follows summation convention is used, i.e., sum on repeated indices is applied.

The principle of complementary virtual work for our system can be stated in the form (Santos *et al.*, 1990a):

$$\int_{\Omega} \delta W^* dx = \int_{\partial\Omega} (\delta f_i^s u_i^s + \delta f_i^n u_i^n + \delta f_i^w u_i^w) d\sigma - \int_{\Omega} [\delta\lambda (\Delta P_n - \Delta P_w - \Delta P_{ca}) + \lambda (\delta\Delta P_n - \delta\Delta P_w - \delta\Delta P_{ca})] dx, \quad (2.9)$$

where  $\delta$  denotes virtual changes and  $\lambda$  is a Lagrange multiplier used to introduce the capillary relation as a constraint. Neglecting second order terms in the different increments and using Eq. (2.6), it can be shown that

$$\begin{aligned} \delta f_i^n &= -\bar{\phi} \left[ \bar{S}_n \delta\Delta P_n + \bar{P}_n \frac{\delta\Delta P_{ca}}{P'_{ca}(\bar{S}_n)} \right] \delta_{ij} v_j, & \delta f_i^w &= -\bar{\phi} \left[ \bar{S}_w \delta\Delta P_w + \bar{P}_w \frac{\delta\Delta P_{ca}}{P'_{ca}(\bar{S}_n)} \right] \delta_{ij} v_j, \\ \delta f_i^s &= -\bar{\phi} \left[ \bar{S}_n \delta\Delta P_n + \bar{S}_w \delta\Delta P_w + P_{ca}(\bar{S}_n) \frac{\delta\Delta P_{ca}}{P'_{ca}(\bar{S}_n)} \right] \delta_{ij} v_j + \delta\Delta\tau_{ij} v_j. \end{aligned} \quad (2.10)$$

Transforming the surface integral in Eq. (2.9) into a volume integral by means of Gauss theorem, using Eqs. (2.10), and following the argument given by Santos *et al.* (1990a) it can be shown that

$$\delta W^* = \epsilon_{ij} \delta\Delta\tau_{ij} + (\bar{S}_n \xi^n - \lambda) \delta\Delta P_n + (\bar{S}_w \xi^w + \lambda) \delta\Delta P_w + (\beta \xi^n + \lambda + (\xi^n - \xi^w) \zeta) \delta\Delta P_{ca}, \quad (2.11)$$

where  $\beta = P_{ca}(\bar{S}_n)/P'_{ca}(\bar{S}_n)$ ,  $\zeta = \bar{P}_w/P'_{ca}(\bar{S}_n)$ . Assuming that  $\delta W^*$  is an exact differential of the variables  $\Delta\tau_{ij}, \Delta P_n, \Delta P_w$ , and  $\Delta P_{ca}$ , the complementary strain energy  $W^*$  is equal to the strain energy  $W$ , and using the capillary relation (2.5) in (2.11) we obtain

$$\begin{aligned}
W &= \frac{1}{2} \{ \varepsilon_{ij} \Delta \tau_{ij} + \xi^n [(\bar{S}_n + \beta + \zeta) \delta \Delta P_n - (\beta + \zeta) \Delta P_w] + \xi^w [(\bar{S}_w + \zeta) \Delta P_w - \zeta \Delta P_n] \} \\
&= \frac{1}{2} \{ \Delta \tau_{ij} \varepsilon_{ij} + [(\bar{S}_n + \beta + \zeta) \xi^n - \zeta \xi^w] \Delta P_n + [(\bar{S}_w + \zeta) \xi^w - (\beta + \zeta) \xi^n] \Delta P_w \}.
\end{aligned} \tag{2.12}$$

Restricting the analysis to the linear range, for the isotropic elastic case we obtain the following stress-strain relations:

$$\begin{aligned}
\text{i)} \quad \Delta \tau_{ij} &= 2N \varepsilon_{ij} + \delta_{ij} (\lambda_c e_b - B_1 \xi^n - B_2 \xi^w), \\
\text{ii)} \quad \Delta \mathcal{T}_n &= (\bar{S}_n + \beta + \zeta) \Delta P_n - (\beta + \zeta) \Delta P_w = -B_1 e_b + M_1 \xi^n + M_3 \xi^w, \\
\text{iii)} \quad \Delta \mathcal{T}_w &= (\bar{S}_w + \zeta) \Delta P_w - \zeta \Delta P_n = -B_2 e_b + M_3 \xi^n + M_2 \xi^w,
\end{aligned} \tag{2.13}$$

where the magnitudes  $\Delta \tau_{ij}$ ,  $\Delta \mathcal{T}_n$  and  $\Delta \mathcal{T}_w$  represent the generalized forces for our system,  $\varepsilon_{ij}$  denotes the strain tensor, and  $e_b = \varepsilon_{ii} = \Delta V_b / \bar{V}_b$ . According to Gassmann (1951), we take  $N$  identical to the shear modulus of the dry rock  $N_m$ . This is also in agreement with the result obtained by Berryman (1999) and implies that the viscous fluids do not transmit shear deformations to the solid. Also,  $\lambda_c = K_c - \frac{2}{3}N$  in 3D and  $\lambda_c = K_c - N$  in 2D, with  $K_c = C_c^{-1}$  being the undrained or closed bulk modulus, computed as in Santos *et al.* (1990b):

$$\begin{aligned}
K_c &= K_s (K_m + \Xi) / (K_s + \Xi), \quad \Xi = K_f (K_m - K_s) / (\bar{\phi} (K_f - K_s)), \\
K_f &= \alpha (\gamma \bar{S}_n C_n + \bar{S}_w C_w)^{-1}, \quad \alpha = 1 + (\bar{S}_n + \beta)(\gamma - 1), \\
\gamma &= (1 + P'_{ca} (\bar{S}_n) \bar{S}_n \bar{S}_w C_w) (1 + P'_{ca} (\bar{S}_n) \bar{S}_n \bar{S}_w C_n)^{-1},
\end{aligned} \tag{2.14}$$

where  $K_m = C_m^{-1}$  and  $K_s = C_s^{-1}$  denote the bulk modulus of the empty matrix and the solid grains, respectively. In the equations above,  $K_f = C_f^{-1}$  defines an effective bulk modulus for a two phase fluid taking into account capillary forces. Here we are assuming that the solid matrix is composed of a homogeneous material. In the case of mixed mineralogy an effective average coefficient for  $K_s$  can be used (see Mavko *et al.*, 1998). The remaining coefficients can be obtained by using the following relations:

$$\begin{aligned}
B_1 &= \chi K_c [(\bar{S}_n + \beta) \gamma - \beta + (\gamma - 1) \zeta], \quad B_2 = \chi K_c [(\bar{S}_w + (1 - \gamma) \zeta)], \\
M_1 &= -M_3 - B_1 C_m \eta^{-1}, \quad M_2 = (r B_2 + \zeta) q^{-1}, \quad M_3 = -M_2 - B_2 C_m \eta^{-1},
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
\chi &= [\eta + \bar{\phi} (C_m - C_c)] \{ \alpha [\eta + \bar{\phi} (C_m - C_f)] \}^{-1}, \quad q = \bar{\phi} (C_n + 1/P'_{ca} (\bar{S}_n) \bar{S}_n \bar{S}_w), \\
r &= (\bar{S}_n + \beta) C_s + (C_c - C_m) [q B_2 + (\bar{S}_n + \beta) (1 - C_s C_c^{-1})], \quad \eta = C_s - C_m.
\end{aligned}$$

The inverse relations of (2.13) giving the generalized strains  $\varepsilon_{ij}$ ,  $(\bar{S}_n + \beta + \zeta) \xi^n - \zeta \xi^w$ , and  $(\bar{S}_w + \zeta) \xi^w - (\beta + \zeta) \xi^n$  as linear functions of  $\Delta \tau_{ij}$ ,  $\Delta P_n$ , and  $\Delta P_w$  are:

$$\begin{aligned}
\text{i)} \quad \varepsilon_{ij} &= 1/(2N) \Delta \tau_{ij} + \delta_{ij} (D \Delta \tau - F_1 \Delta P_n - F_2 \Delta P_w), \\
\text{ii)} \quad (\bar{S}_n + \beta + \zeta) \xi^n - \zeta \xi^w &= -F_1 \Delta \tau + H_1 \Delta P_n + H_3 \Delta P_w, \\
\text{iii)} \quad (\bar{S}_w + \zeta) \xi^w - (\beta + \zeta) \xi^n &= -F_2 \Delta \tau + H_3 \Delta P_n + H_2 \Delta P_w.
\end{aligned} \tag{2.16}$$

In the equations above  $\Delta\tau = \Delta\tau_{ii}$  and  $D, F_1, F_2, H_1, H_2, H_3$  are elastic coefficients (Santos *et al.*, 1990a).

2.2. Definition of bulk and pore compressibilities

Here we show that the model presented above enables us to establish a theory for the compressibilities of the system. For the analysis that follows we consider that the variable  $P_c$  (applied external pressure) plays the role of the confining or overburden pressure supported by the porous saturated sample at a given (fixed) depth.

First we proceed to derive expressions for  $e_b$  and  $e_p = \Delta V_p / \bar{V}_p$  that will be used to define the compressibilities for the saturated material. Consider tensional changes  $\Delta\tau_{ij}$  such that  $\Delta\tau_{ij} = 0$  for  $i \neq j$  and

$$\Delta\tau_{11} = \Delta\tau_{22} = \Delta\tau_{33} = \Delta\tau/3 = -\Delta P_c, \quad \Delta P_c > 0. \tag{2.17}$$

Note that from Eqs. (2.2), (2.4) and (2.16i) we can express  $e_b$  and  $e_p$  in the form:

$$e_b = (3D + 1/2N)\Delta\tau - 3F_1\Delta P_n - 3F_2\Delta P_w \tag{2.18}$$

$$e_p = \xi^*/\bar{\phi} + \Delta V_p^c/\bar{V}_p = \xi^*/\bar{\phi} - \bar{S}_n C_n \Delta P_n - \bar{S}_w C_w \Delta P_w \tag{2.19}$$

Next, adding (2.16ii) and (2.16iii) we obtain

$$\xi^* = -(F_1 + F_2)\Delta\tau + (H_1 + H_3)\Delta P_n + (H_2 + H_3)\Delta P_w. \tag{2.20}$$

Using the relations  $(3D + 1/2N) = C_m/3$ ,  $F_1 = (\bar{S}_n + \beta)\eta/3$ ,  $F_2 = (\bar{S}_w - \beta)\eta/3$ , [see Santos *et al.* (1990b), pp. 1432] and combining Eqs. (2.17), (2.20) and the capillary relation  $\Delta P_n - \Delta P_w = \Delta P_{ca}$ , Eqs. (2.18) and (2.19) become

$$e_b = -C_m \Delta P_c - \eta \Delta P_w - (\bar{S}_n + \beta) \eta \Delta P_{ca}, \tag{2.21}$$

$$e_p = (\eta/\bar{\phi})\Delta P_c + ([H_1 + H_2 + 2H_3]/\bar{\phi} - \bar{S}_n C_n - \bar{S}_w C_w)\Delta P_w + ([H_1 + H_3]/\bar{\phi} - \bar{S}_n C_n)\Delta P_{ca}. \tag{2.22}$$

Since there are two volumes  $V_b$  and  $V_p$  and three different pressures ( $P_c, P_w, P_{ca}$ ) that may vary independently, from (2.21)-(2.22) we see that a set of six compressibilities can be defined as:

$$C_{bc} = -\frac{1}{V_b} \left( \frac{\Delta V_b}{\Delta P_c} \right)_{P_w, P_{ca}} = C_m, \tag{2.23}$$

$$C_{bw} = -\frac{1}{V_b} \left( \frac{\Delta V_b}{\Delta P_w} \right)_{P_c, P_{ca}} = C_m - C_s, \tag{2.24}$$

$$C_{bca} = -\frac{1}{V_b} \left( \frac{\Delta V_b}{\Delta P_{ca}} \right)_{P_c, P_w} = (\bar{S}_n + \beta)(C_m - C_s), \tag{2.25}$$

$$C_{pc} = -\frac{1}{V_p} \left( \frac{\Delta V_p}{\Delta P_c} \right)_{P_w, P_{ca}} = \frac{1}{\bar{\phi}}(C_m - C_s), \tag{2.26}$$

$$C_{pw} = \frac{1}{\bar{V}_p} \left( \frac{\Delta V_p}{\Delta P_w} \right)_{P_c, P_{ca}} = [H_1 + H_2 + 2H_3] / \bar{\phi} - \bar{S}_n C_n - \bar{S}_w C_w, \quad (2.27)$$

$$C_{pca} = \frac{1}{\bar{V}_p} \left( \frac{\Delta V_p}{\Delta P_{ca}} \right)_{P_c, P_w} = [H_1 + H_3] / \bar{\phi} - \bar{S}_n C_n, \quad (2.28)$$

Then, Eqs. (2.21) and (2.22) can be stated in differential form as follows:

$$e_b = -C_{bc}(P_c, P_w, P_{ca})dP_c + C_{bw}(P_c, P_w, P_{ca})dP_w + C_{bca}(P_c, P_w, P_{ca})dP_{ca} \quad (2.29)$$

$$e_p = -C_{pc}(P_c, P_w, P_{ca})dP_c + C_{pw}(P_c, P_w, P_{ca})dP_w + C_{pca}(P_c, P_w, P_{ca})dP_{ca} \quad (2.30)$$

The coefficients  $C_{bc}$ ,  $C_{bw}$  and  $C_{pc}$  are similar to those defined in Zimmerman *et al.* (1986) and Zimmerman (1991) for the case of single-phase fluids with analogous interpretation and applications. The remaining coefficients  $C_{bca}$ ,  $C_{pw}$  and  $C_{pca}$  are strongly dependent on the saturation state and capillary pressure. This means that the changes in such variables can also modify the pore and bulk volumes.

Next note that from Eqs. (2.23), (2.24) and (2.26) we deduce that

$$\text{i) } C_{bw} = C_{bc} - C_s, \quad \text{ii) } C_{bw} = \bar{\phi} C_{pc}. \quad (2.31)$$

Eqs. (2.31i) and (2.31ii) correspond to Eqs. (2.5) and (2.6) in Zimmerman (1991). This shows the equivalence of the variable  $P_w$  and the ‘‘pore pressure’’  $P_p$  used in such reference and that  $C_{bw}$  plays the role of the compressibility  $C_{bp}$  in Zimmerman *et al.* (1986) and in Zimmerman (1991).

### 3. Effective pressure law for elastic bulk volume deformations

The infinitesimal pore and bulk strains found in the preceding subsection correspond to pressure changes small enough so that the different compressibilities involved can be considered independent of pressures during the process. That analysis would be appropriate for elastic wave propagation problems. However, for other physical problems, the total integrated strain is required (Zimmerman, 1991). Examples of such situations are the studies of subsidence and formation compaction due to pore fluid migration from hydrocarbon or groundwater reservoirs, the evaluation of the increase (decrease) in the pore volume of a rock due to a finite increase (decrease) in the pressure of the pore fluids and the estimation of the true in-situ porosity of a formation from laboratory measurements (usually made at zero confining pressure). It is a well established fact that most of the mechanical and transport properties of cracked or porous media subjected to both external and internal pressures depend on effective pressure. This is a very important concept since the forementioned properties remain the same at a constant effective pressure, even though the confining and pore fluid pressures are varied. Important contributions to the formulation and analysis of effective stress rules, from both the theoretical and experimental points of view, were presented by different authors, as for example Terzaghi (1936), Geertsma (1957), Nur and Byerlee (1971), Christensen and Wang (1985), Zimmerman (1991), Berryman (1992), Gangi and Carlson (1996) and Prasad and Manghnani (1997). Next we will

show that the equations obtained in section 2.2. lead us to an extension of the effective pressure coefficients obtained by Zimmerman (1991) for uniform pore pressure systems, when capillary forces are considered. Assuming that  $e_b$  and  $e_p$  are exact differentials of the variables  $P_c$ ,  $P_w$  and  $P_{ca}$  and that  $C_s$  is constant, from Eqs. (2.29) and (2.24) we get

$$\frac{\partial}{\partial P_w} \left[ \frac{\partial e_b}{\partial P_c} \right] = - \frac{\partial C_{bc}}{\partial P_w}, \tag{3.1}$$

$$\frac{\partial}{\partial P_c} \left[ \left( \frac{\partial e_b}{\partial P_w} \right) \right] = \frac{\partial C_{bw}}{\partial P_c} = \frac{\partial [C_{bc} - C_s]}{\partial P_c} = \frac{\partial C_{bc}}{\partial P_c}. \tag{3.2}$$

Consequently  $C_{bc} = C_{bc}(P_c - P_w, P_{ca})$ , and from Eq. (2.25) we also have that

$$C_{bca} = C_{bca}(P_c - P_w, P_{ca}) = (\bar{S}_n + \beta)(C_{bc}(P_c - P_w, P_{ca}) - C_s). \tag{3.3}$$

Next, following Zimmerman (1991), to obtain the total strain through any given process we integrate relations (2.29) and (2.30). To perform the integration in the  $(P_c, P_w, P_{ca})$  state variables, we move along the following paths:

- Path 1:  $(0, 0, P_{ca}(S_n^*)) \rightarrow (P_c, 0, P_{ca}(S_n^*))$
- Path 2:  $(P_c, 0, P_{ca}(S_n^*)) \rightarrow (P_c, \bar{P}_w, P_{ca}(S_n^*))$
- Path 3:  $(P_c, \bar{P}_w, P_{ca}(S_n^*)) \rightarrow (P_c, \bar{P}_w, P_{ca}(\bar{S}_n))$ .

The symbol  $S_n^*$  denotes an unstrained reference saturation state within the range  $S_{rn} < S_n^* < 1 - S_{rw}$ . For the Path 1, the sample is subjected to an external pressure  $P_c$  and the wetting fluid pressure is held at  $P_w = 0$ , while the non-wetting fluid is held at the reference pressure  $\bar{P}_n = P_{ca}(S_n^*)$  so that both pressures are constant during the experiment. Then from Eq. (2.29) the integrated strain  $E_b^1$  along this path results

$$E_b^1 = - \int_0^{P_c} C_{bc}(P_1 - 0, P_{ca}(S_n^*)) dP_1$$

Next, for the Path 2, the external pressure  $P_c$  is held constant while both fluids receive an equal increment in pressure from 0 to  $\bar{P}_w$ . In this way there is no change in capillary pressure. The resulting bulk deformation during this process  $E_b^2$  is

$$E_b^2 = \int_0^{\bar{P}_w} [C_{bc}(P_c - P_2, P_{ca}(S_n^*)) - C_s] dP_2 = - \int_{P_c}^{P_c - \bar{P}_w} C_{bc}(P, P_{ca}(S_n^*)) dP - C_s \bar{P}_w.$$

Finally for the Path 3 experiment the wetting fluid pressure is held at the pressure  $\bar{P}_w$  while the non-wetting fluid pressure is incremented from  $P_{ca}(S_n^*)$  to  $P_{ca}(\bar{S}_n)$ . This results in an increment in capillary pressure with constant  $\bar{P}_w$  and  $P_c$  with bulk volume deformation  $E_b^3$ :

$$E_b^3 = \int_{P_{ca}(S_n^*)}^{P_{ca}(\bar{S}_n)} C_{bca}(P_c - \bar{P}_w, P_3) dP_3.$$



The total strain  $E_b$  is then given by the sum:  $E_b = E_b^1 + E_b^2 + E_b^3$ .

Next, following Zimmerman (1991), we proceed to define the “secant” compressibilities  $\bar{C}_{bc}$  and  $\bar{C}_{bca}$  to obtain an effective stress form of the total strain  $E_b$ . Set  $\mathcal{R}(P_{ca}) = P_{ca}(\bar{S}_n) - P_{ca}(S_n^*)$  and let

$$\begin{aligned} \bar{C}_{bc}(P_c, P_{ca}(\bar{S}_n)) &= \frac{1}{P_c} \int_0^{P_c} C_{bc}(P, P_{ca}(\bar{S}_n)) dP, \\ \bar{C}_{bca}(P_c - \bar{P}_w, P_{ca}(S_n^*), P_{ca}(\bar{S}_n)) &= \frac{1}{\mathcal{R}(P_{ca})} \int_{P_{ca}(S_n^*)}^{P_{ca}(\bar{S}_n)} C_{bca}(P_c - \bar{P}_w, P_3) dP_3 \\ &= \frac{1}{\mathcal{R}(P_{ca})} \int_{S_n^*}^{\bar{S}_n} [s + \beta(s)] [C_{bc}(P_c - \bar{P}_w, P_{ca}(s)) - C_s] P_{ca}'(s) ds, \end{aligned}$$

where we have used Eq. (2.25) and the fact that  $P_3 = P_3(s)$  where the variable  $s$  takes values in the saturation range  $[S_n^*, \bar{S}_n]$  and  $P_3(s)$  is the capillary pressure function. Then we see that the elastic deformation  $E_b$  of the bulk volume can be written in terms of an effective pressure  $P_{ef}^b$  in the form

$$\begin{aligned} E_b \approx & -\bar{C}_{bc}(P_c - \bar{P}_w, P_{ca}(S_n^*)) (P_c - \bar{P}_w) - C_s \bar{P}_w \\ & + \bar{C}_{bca}(P_c - \bar{P}_w, P_{ca}(S_n^*), P_{ca}(\bar{S}_n)) \mathcal{R}(P_{ca}) = -\bar{C}_{bc}(P_c - \bar{P}_w, P_{ca}(S_n^*)) P_{ef}^b, \end{aligned} \tag{3.4}$$

where

$$P_{ef}^b = P_c - \bar{n}_{b1} \bar{P}_w - \bar{n}_{b2} \mathcal{R}(P_{ca}), \tag{3.5}$$

$$\bar{n}_{b1} = 1 - C_s / \bar{C}_{bc}(P_c - \bar{P}_w, P_{ca}(S_n^*)), \tag{3.6}$$

$$\bar{n}_{b2} = \bar{C}_{bca}(P_c - \bar{P}_w, P_{ca}(S_n^*), P_{ca}(\bar{S}_n)) / \bar{C}_{bc}(P_c - \bar{P}_w, P_{ca}(S_n^*)). \tag{3.7}$$

It is therefore seen that, the effect of the confining pressure is not only counteracted by the wetting fluid pressure (as may be expected) but also by a new term related to the capillary pressure change during the deformation process. A similar expression can be obtained for the total elastic deformation  $P_{ef}^p$  of the pore volume  $V_p$ . We do not include it here for brevity.

#### 4. The equations of motion

First we will get an expression for the complementary potential energy density  $\mathcal{V}_d^*$  of the system. Consider a perturbation of the system from the equilibrium state. According to Eq. (2.8)

$$\delta \mathcal{V}^* = \int_{\Omega} \delta \mathcal{V}_d^* dx = \int_{\Omega} \delta W^* dx - \int_{\partial \Omega} (u_i^s \delta f_i^s + \tilde{u}_i^n \delta f_i^n + \tilde{u}_i^w \delta f_i^w) d\sigma. \tag{4.1}$$

Using Eqs. (2.10) and (2.11), we obtain

$$\delta V^* = \int_{\Omega} \left\{ -u_i^s \delta \frac{\partial \Delta \tau_{ij}}{\partial x_j} + u_i^n \delta \frac{\partial}{\partial x_i} \delta \Delta T_n + u_i^w \delta \frac{\partial}{\partial x_i} \delta \Delta T_w + \lambda \delta [\Delta P_{ca} - (\Delta P_n - \Delta P_w)] \right\} dx.$$

As stated previously, since we are in the linear case, the complementary strain energy  $W^*$  equals the strain energy  $W$  and consequently the same holds for their corresponding densities  $V_d^* = V_d$ , (Fung, 1965; Santos *et al.*, 1990a). Then, assuming that  $V_d$  is an exact differential we have

$$V_d = -u_i^s \frac{\partial \Delta \tau_{ij}}{\partial x_j} + u_i^n \frac{\partial \Delta T_n}{\partial x_i} + u_i^w \frac{\partial \Delta T_w}{\partial x_i} + \lambda (\Delta P_c - (\Delta P_n - \Delta P_w)). \tag{4.2}$$

Thus, if  $u = (u_i^s, u_i^n, u_i^w) = (u_j)$ ,  $1 \leq i \leq 3$ ,  $1 \leq j \leq 9$ , are chosen as generalized coordinates and assuming that the system is conservative we get

$$\frac{\partial V_d}{\partial u_i^s} = -\frac{\partial \Delta \tau_{ij}}{\partial x_j}, \quad \frac{\partial V_d}{\partial u_i^n} = \frac{\partial \Delta T_n}{\partial x_i}, \quad \frac{\partial V_d}{\partial u_i^w} = \frac{\partial \Delta T_w}{\partial x_i}, \quad 1 \leq i \leq 3 \tag{4.3}$$

Next, following Santos *et al.* (1990a), in the isotropic case the kinetic energy density  $E_c$  and the dissipation density function  $\mathcal{D}$  have the form

$$E_c = \frac{1}{2} \rho \frac{\partial u_i^s}{\partial t} \frac{\partial u_i^s}{\partial t} + \rho_n S_n \frac{\partial u_i^n}{\partial t} \frac{\partial u_i^n}{\partial t} + \rho_w S_w \frac{\partial u_i^w}{\partial t} \frac{\partial u_i^w}{\partial t} \tag{4.4}$$

$$E_c + \frac{1}{2} g_n^c \frac{\partial u_i^n}{\partial t} \frac{\partial u_i^n}{\partial t} + \frac{1}{2} g_w^c \frac{\partial u_i^w}{\partial t} \frac{\partial u_i^w}{\partial t} + g_{nw}^c \frac{\partial u_i^n}{\partial t} \frac{\partial u_i^w}{\partial t},$$

$$\mathcal{D}_c = \frac{1}{2} (d_n^c \frac{\partial u_i^n}{\partial t} \frac{\partial u_i^n}{\partial t} + d_w^c \frac{\partial u_i^w}{\partial t} \frac{\partial u_i^w}{\partial t} - d_{nw}^c \frac{\partial u_i^n}{\partial t} \frac{\partial u_i^w}{\partial t}), \tag{4.5}$$

where  $\rho_w$  and  $\rho_n$  are the mass densities of the wetting and the non-wetting fluids and  $\rho$  is the density of the bulk material, i.e.,  $\rho = (1 - \bar{\phi})\rho_s + \bar{S}_n \rho_n + \bar{S}_w \rho_w$ , with  $\rho_s$  being the mass density of the solid grains. Also, the mass coupling coefficients  $g_n^c, g_w^c, g_{nw}^c$  represent the inertial effects associated with dynamic interactions among the three phases, while the coefficients  $d_n^c, d_w^c, d_{nw}^c$  include the viscous coupling between the solid and fluid phases. They can be computed by the formulae (see Santos *et al.*, 1990a, 1990b)

$$g_l^c = G \rho_l \bar{S}_l / \bar{\phi}, \quad d_l^c = \mu_l (\bar{S}_l)^2 \mathcal{A}_l, \quad l = n, w, \tag{4.6}$$

$$g_{nw}^c = \epsilon G (\rho_n \rho_w \bar{S}_n \bar{S}_w)^{\frac{1}{2}} / \bar{\phi}, \quad d_{nw}^c = (\mu_n \mu_w)^{\frac{1}{2}} \bar{S}_n \bar{S}_w K_{r_{nw}} / \mathcal{A}, \tag{4.7}$$

The factor  $G$  is known as a structure factor and is related to the tortuosity of the pore space; it can be estimated as follows (Berryman, 1981):  $G = \frac{1}{2} (1 + \frac{1}{\bar{\phi}})$ . The constants  $\mu_n, \mu_w$  are the non-wetting and wetting fluid viscosities, respectively. Also,  $\mathcal{A} = K (K_{rn} K_{rw} - K_{r_{nw}}^2)$ ,  $\mathcal{A}_n = K_{rw} / \mathcal{A}$ ,  $\mathcal{A}_w = K_{rn} / \mathcal{A}$ , with  $K, K_{rn}, K_{rw}$  and  $K_{r_{nw}}$  denoting the absolute and relative permeabilities, respectively. The

relative permeabilities are assumed to be functions of the non-wetting saturation  $S_n$  (Bear, 1972; Scheidegger, 1974; Peaceman, 1977).

**REMARK.** Note that Eq. (4.5) contains an additional viscous coupling coefficients  $d_{nw}^c$  taking into account the viscous drag between the immiscible fluids. This effect was not present in the original formulation in Santos *et al.* (1990a).

Finally, the Lagrange formulation of the equations of motion is given by

$$\frac{d}{dt} \left( \frac{\partial E_c}{\partial \dot{u}_j} \right) + \frac{\partial \mathcal{D}}{\partial \dot{u}_j} = - \frac{\partial \mathcal{V}_d}{\partial u_j}, \quad 1 \leq j \leq 9 \tag{4.8}$$

### 5. Inclusion of viscoelastic dissipation

So far, the only source of energy dissipation considered in the equations of motion is associated with the viscous solid-fluid interaction (global flow effect) given by Eq. (4.5), since the solid phase was assumed to be perfectly elastic. However, it is well known that wave dispersion and attenuation phenomena in real saturated rocks are higher than those associated with viscodynamic effects (Stoll and Bryan, 1970; Mochizuki, 1982; Carcione, 2001). This is mainly due to the complexity of pore shapes, heterogeneities in the physical properties and in the distribution of the fluids, the intrinsic anelasticity of the frame and any other process associated to solid-fluid interactions. These factors can be included in the formulation by means of the theory of viscoelasticity. The theoretical basis for this generalization was given by Biot (1956a, 1962), who developed the general theory of deformation of porous materials saturated by viscous fluids when the solid phase exhibits linear viscoelastic behaviour. Using principles of irreversible thermodynamics he established a general operational relationship between generalized forces  $Q_i$  and observed coordinates  $q_i$ , of the form  $Q_i = T_{ij}q_j$ , where  $T_{ij}$  is a symmetric matrix. In this way Biot obtained a general correspondence rule between the elastic and viscoelastic formulations in the domain of the Laplace transform and showed that formally all the relations are identical. The poroviscoelastic formulation obtained in this way was later applied by different authors for the study of wave propagation problems (see Stoll and Bryan, 1970; Stoll, 1974; Keller, 1989; Rasolofosaon, 1991). It follows from Eqs. (2.13) that the eight generalized forces of our model are related to the variables  $\xi^n$ ,  $\xi^w$  and  $\epsilon_{ij}$  by means of a symmetric matrix, whose elements are functions of the elastic coefficients. Thus, if we assume that the solid phase shows linear viscoelastic behaviour, we are able to extend the constitutive relations (2.13) by simply replacing the real elastic moduli  $N$ ,  $\lambda_c$ ,  $B_1$ ,  $B_2$ ,  $M_1$ ,  $M_2$ ,  $M_3$  by appropriate viscoelastic operators. Using Fourier transform in time (instead of Laplace), we can state these relations in the space frequency domain as follows:

$$\begin{aligned} \text{i)} \quad \Delta \widehat{\tau}_{ij}(\omega) &= 2\widehat{N}(\omega)\widehat{\epsilon}_{ij}(\omega) + \delta_{ij}[\widehat{\lambda}_c(\omega)\widehat{e}_b(\omega) - \widehat{B}_1(\omega)\widehat{\xi}^n(\omega) - \widehat{B}_2(\omega)\widehat{\xi}^w(\omega)], \\ \text{ii)} \quad \Delta \widehat{\mathcal{T}}_n(\omega) &= -\widehat{B}_1(\omega)\widehat{e}_b(\omega) + \widehat{M}_1(\omega)\widehat{\xi}^n(\omega) + \widehat{M}_2(\omega)\widehat{\xi}^w(\omega), \\ \text{iii)} \quad \Delta \widehat{\mathcal{T}}_w(\omega) &= -\widehat{B}_2(\omega)\widehat{e}_b(\omega) + \widehat{M}_3(\omega)\widehat{\xi}^n(\omega) + \widehat{M}_2(\omega)\widehat{\xi}^w(\omega), \end{aligned} \tag{5.1}$$

where  $\omega = 2\pi f$  is the angular frequency,  $\widehat{N}$ ,  $\widehat{\lambda}_c$ ,  $\widehat{B}_1$ ,  $\widehat{B}_2$ ,  $\widehat{M}_1$ ,  $\widehat{M}_2$ ,  $\widehat{M}_3$  are complex frequency dependent poroviscoelastic moduli and the hat denotes time Fourier transform. A similar extension can be done for the strain-stress relations (2.16).

By definition, viscoelastic behaviour is characterized by the time dependent relationship between stresses and strains (memory effect). This can be clearly appreciated formulating Eqs. (5.1) in the time domain. To obtain expressions analogous to the Boltzmann's integral formulation in viscoelasticity, first we need to introduce the stress relaxation functions  $R_J(t)$ ,  $J = N, \lambda_c, B_1, B_2, M_1, M_2, M_3$ , whose time derivatives  $\dot{R}_J(t)$  are related to the complex moduli in Eqs. (5.1) as stated in Eq. (5.3) below. These relaxation functions characterize the response of the system when step-like strains are applied. In order to preserve the causality of the model it is necessary to assume that all the relaxation functions are zero for times  $t < 0$ , so that the stress at time  $t$  cannot be influenced by future strains. Taking this into account in the inverse Fourier integrals of Eqs. (5.1) and integrating by parts assuming that  $\varepsilon_{ij}(-\infty) = \xi^n(-\infty) = \xi^w(-\infty) = 0$ , Eqs. (5.1) can be stated in the space-time domain as follows:

$$\begin{aligned}
 \text{i) } \Delta\tau_{ij}(t) &= 2R_N(0+)\varepsilon_{ij}(t) + \delta_{ij}[R_{\lambda_c}(0+)e_b(t) - R_{B_1}(0+)\xi^n(t) - R_{B_2}(0+)\xi^w(t)] \\
 &+ 2\int_0^\infty \dot{R}_N(s)\varepsilon_{ij}(t-s)ds + \delta_{ij}\int_0^\infty [\dot{R}_{\lambda_c}(s)e_b(t-s) - \dot{R}_{B_1}(s)\xi^n(t-s) - \dot{R}_{B_2}(s)\xi^w(t-s)]ds, \\
 \text{ii) } \Delta\mathcal{T}_n(t) &= -R_{B_1}(0+)e_b(t) + R_{M_1}(0+)\xi^n(t) + R_{M_3}(0+)\xi^w(t) \\
 &+ \int_0^\infty [-\dot{R}_{B_1}(s)e_b(t-s) + \dot{R}_{M_1}(s)\xi^n(t-s) + \dot{R}_{M_3}(s)\xi^w(t-s)]ds, \tag{5.2} \\
 \text{iii) } \Delta\mathcal{T}_w(t) &= -R_{B_2}(0+)e_b(t) + R_{M_3}(0+)\xi^n(t) + R_{M_2}(0+)\xi^w(t) \\
 &+ \int_0^\infty [-\dot{R}_{B_2}(s)e_b(t-s) + \dot{R}_{M_3}(s)\xi^n(t-s) + \dot{R}_{M_2}(s)\xi^w(t-s)]ds.
 \end{aligned}$$

In general, the notation  $R_J(0+)$  denotes  $\lim_{t \rightarrow 0^+} R_J(t)$ . Let  $\dot{F}^S(\omega)$  and  $\dot{F}^C(\omega)$  be the half-range Fourier sine and cosine transforms of the function  $\dot{f}(t)$ . Then, the following frequency-domain relations between complex moduli and relaxation functions can be shown:

$$\widehat{J}(\omega) = \widehat{J}_R(\omega) + i\widehat{J}_I(\omega) = R_J(0^+) + \dot{R}_J^C(\omega) - i\dot{R}_J^S(\omega), \tag{5.3}$$

where  $J$  stands for  $N, \lambda_c, B_1, B_2, M_1, M_2, M_3$ .

As pointed out in Zimmerman (1991), although the derivatives appearing in the different compressibilities (2.23)-(2.28) can be also defined for irreversible deformation processes, such coefficients are generally computed only for purely elastic cases. Then for consistency, the compressibilities of the model could be extended to the present case by considering the relaxed limits (i.e.  $\lim \omega \rightarrow 0$  or  $\lim t \rightarrow \infty$ ) of the complex moduli appearing in Eq. (5.3). In the Appendix we give a set of restrictions imposed by the Laws of Thermodynamics on the imaginary parts of the coefficients in Eqs. (5.1).

### 6. The equations of motion for the full frequency range

It is known that, for single phase fluids, the viscous and mass coupling coefficients become frequency dependent in the high frequency range (see Biot, 1956c; Johnston *et al.*, 1987, Carcione, 2001). This effect is associated with the departure of the flow from the laminar

Poiseuille type at the pore scale, which occurs for frequencies greater than some characteristic value. Since we have two immiscible fluids flowing within the poral space and three relative permeability functions, following the ideas in Berryman *et al.* (1988), we define for this model three characteristic frequencies of the form

$$\omega_c^l = \bar{S}_l \mu_l \bar{\phi} \mathcal{A}_l / (G \rho_l), \quad l = n, w, \tag{6.1}$$

$$\omega_c^{nw} = \bar{\phi} (\mu_n \mu_w)^{\frac{1}{2}} (\bar{S}_n \bar{S}_w)^{\frac{1}{2}} K_{rw} / (\epsilon \mathcal{A} G (\rho_n \rho_w)^{\frac{1}{2}}). \tag{6.2}$$

These frequencies are obtained by equating the inertial and viscous drag terms associated with each fluid phase and using Eqs. (4.6) and (4.7). For frequencies bigger than the minimum of the three characteristic frequencies defined above, the mass and viscous coupling coefficients are defined as follows:

$$g_l(\omega) = g_l^c + \mu_l (\bar{S}_l)^2 \mathcal{A}_l F_l(\theta_l) / \omega, \quad l = n, w, \tag{6.3}$$

$$g_{nw}(\omega) = g_{nw}^c + (\mu_n \mu_w)^{\frac{1}{2}} \bar{S}_n \bar{S}_w K_{rw} F_l(\theta_{nw}) / (\omega \mathcal{A}), \tag{6.4}$$

$$d_l(\omega) = d_l^c F_R(\theta_l) \quad l = n, w, nw, \tag{6.5}$$

The complex valued frequency dependent function  $F(\theta_j) = F_R(\theta_j) + iF_I(\theta_j)$ ,  $j = n, w, nw$  in Eqs. (6.3)-(6.5) is the “universal” frequency correction function defined by Biot (1956c) for single phase fluids:

$$F(\theta) = \frac{1}{4} \frac{\theta T(\theta)}{1 - \frac{2}{i\theta} T(\theta)}, \quad T(\theta) = \frac{\text{ber}'(\theta) + i \text{bei}'(\theta)}{\text{ber}(\theta) + i \text{bei}(\theta)},$$

with  $\text{ber}(\theta)$  and  $\text{bei}(\theta)$  being the Kelvin functions of the first kind and zero order. The arguments  $\theta_j$  for  $F(\theta_j)$ ,  $j = n, w, nw$  in Eqs. (6.3)-(6.5) can be estimated as in Biot (1956c) and Santos *et al.* (1992):

$$\theta_j = a_p^j \sqrt{\omega \rho_j / \mu_j}, \quad a_p^j = 2 \sqrt{K k_{rj} A_0 / \bar{\phi}}, \quad j = n, w, nw \tag{6.6}$$

where  $A_0$  denotes the Kozeny-Carman constant (see Bear 1972; Hovem and Ingram 1979).

Next, combine Eq. (4.8) with Eqs. (4.3), (4.4), (4.5) and (5.1) and assume that  $S_n$  is independent of time [see Santos *et al.* (1990a) for the argument justifying this assumption].

For a spatially homogeneous medium, the Eqs. of motion (4.8) formulated in the space-frequency domain take the following form:

$$\begin{aligned} -\omega^2 (\rho \hat{u}^s + \rho_n \bar{S}_n \hat{u}^n + \rho_w \bar{S}_w \hat{u}^w) &= (\widehat{K}_c + \frac{4}{3} \widehat{N}) \nabla \nabla \cdot \hat{u}^s - \widehat{N} \nabla \times \nabla \times \hat{u}^s + \widehat{B}_1 \nabla (\nabla \cdot \hat{u}^n) + \widehat{B}_2 \nabla (\nabla \cdot \hat{u}^w) \\ -\omega^2 (\rho_n \bar{S}_n \hat{u}^s + g_n \hat{u}^n + g_{nw} \hat{u}^w) + i \omega d_n \hat{u}^n - i \omega d_{nw} \hat{u}^w &= \widehat{B}_1 \nabla (\nabla \cdot \hat{u}^s) + \widehat{M}_1 \nabla (\nabla \cdot \hat{u}^n) + \widehat{M}_3 \nabla (\nabla \cdot \hat{u}^w) \\ -\omega^2 (\rho_w \bar{S}_w \hat{u}^s + g_{nw} \hat{u}^n + g_w \hat{u}^w) + i \omega d_w \hat{u}^w - i \omega d_{nw} \hat{u}^n &= \widehat{B}_2 \nabla (\nabla \cdot \hat{u}^s) + \widehat{M}_3 \nabla (\nabla \cdot \hat{u}^n) + \widehat{M}_2 \nabla (\nabla \cdot \hat{u}^w). \end{aligned} \tag{6.7}$$

Taking into account the fact that  $F_R(\theta) \rightarrow 1$  and  $F_I(\theta)/\omega \rightarrow 0$  as  $\omega \rightarrow 0$ , we may regard Eqs. (6.7) as the general form of the equations of motion for frequencies ranging from the seismic to the ultrasonic range.

The plane wave analysis performed in Santos *et al.* (1990a) shows that in these type of media, three different compressional waves (Type I, Type II and Type III) and one shear wave (or S-wave) can propagate. The Type-I and S waves behave similarly to the classical P and S waves propagating in elastic or viscoelastic isotropic solids and they are associated to the motion of both solid and fluids moving nearly in phase. The Type-II and Type-III are slow waves strongly attenuated in the low frequency range, as will be confirmed in the next section. The first one is analogous to the classical Biot's slow wave [observed in laboratory experiments by Plona (1980)], with the two fluids moving nearly in phase and in opposite phase with the solid. For the Type III waves, both fluids move in nearly opposite phase with each other (Santos *et al.*, 2004). This is a new mode not present in the case of single-phase fluids. The particle displacements associated to the different wave modes propagating in linear poroviscoelastic media saturated by single phase fluids were analyzed rigorously by Rasolofosaon (1991), who demonstrated their elliptical polarization.

Denoting by  $k^j, j = I, II, III, S$ , the complex wavenumbers, the phase velocities  $C$  and quality factors  $Q$  are obtained by using the formulae:

$$C_j = \omega / \text{Re}(k_j), \quad Q_j = -\text{Re}(k_j) / 2\text{Im}(k_j), \tag{6.8}$$

### 7. Application to a real sandstone

We use the model to analyze the combined effects of saturation and effective pressure on the attributes of the different waves propagating in a sample of Boise sandstone, a well consolidated feldspathic graywacke (see Mann and Fatt, 1960; King, 1966). Its material properties are  $\bar{\phi} = 0.25$ ,  $K = 1400 \cdot 10^{-15} \text{ m}^2$ ,  $K_s = 34.5 \text{ GPa}$  and  $\rho_s = 2550 \text{ kg/m}^3$ . To introduce the variation of the matrix properties with effective pressure, we obtained the dependence of the shear and bulk dry-rock moduli  $N_m(P_c)$  and  $K_m(P_c)$  versus confining pressure by assuming that for each  $P_c$  the velocities measured by King (1966) correspond to the elastic isotropic approximation. Then, we performed exponential-type regressions of the form:

$$f^{-1}(P_c) = y_0 + A_1 \exp(-(P_c - P_0)/t_1) + A_2 \exp(-(P_c - P_0)/t_2), \tag{7.1}$$

where  $f$  stands for  $N_m$  or  $K_m$ . The regression coefficients are given in Table 1. Since for the dry sample the effective pressure equals the confining pressure, to obtain the elastic properties of the skeleton for a given combination of confining and fluid pressures, we simply replace  $P_c$  by the effective pressure  $P_{ef}^b$  [given by Eq. (3.5)] in Eq. (7.1). As explained in Section 2.1., the overall shear modulus  $N$  is taken equal to  $N_m$ .

Table 1 - Coefficients of Eq. (7.1) for Boise sandstone. For pressures given in MPa the moduli result in GPa.

	$P_0$	$y_0$	$A_1$	$t_1$	$A_2$	$t_2$
$(K_m)^{-1}$	0.536626645088	0.092089	0.01766	7.634	0.01828	38.53
$(N_m)^{-1}$	4.427347752235	0.10113	0.012	1.925	0.01259	23.71

The capillary pressure function  $P_{ca}(S_n)$  and the relative permeability functions  $K_{rn}(S_n)$  and  $K_{rw}(S_n)$  needed to describe our system are taken to be (see Douglas *et al.*, 1993):

$$P_{ca}(S_n) = A \{ 1 / (S_n + S_{rw} - 1)^2 - S_{rn}^2 / [S_n (1 - S_{rn} - S_{rw})]^2 \} ,$$

$$K_{rn}(S_n) = [1 - (1 - S_n) / (1 - S_{rn})]^2 , \quad K_{rw}(S_n) = [(1 - S_n - S_{rw}) / (1 - S_{rw})]^2 . \quad (7.2)$$

These relations are based on laboratory experiments performed on different porous rocks during imbibition and drainage processes (neglecting hysteresis effects). We chose  $S_{rw} = S_{rn} = 0.05$ , and  $A = 30$  kPa. In the absence of proper experimental data, the coupling permeability function  $K_{rmw}(S_n)$  used in this work is assumed to be  $K_{rmw}(S_n) = \sqrt{\epsilon} K_{rn}(S_n) K_{rw}(S_n)$ . The parameter  $\epsilon$  in Eqs. (4.7) and the definition of  $K_{rmw}$  is equal to 0.1, as in Santos *et al.* (1990b). The whole set of poroelastic moduli is computed using Eqs. (2.14) and (2.15).

The viscoelastic behaviour is introduced using the phenomenological model defined by Liu *et al.* (1976), consisting in a continuous superposition of Zener elements or relaxation times. This results in a linear causal model, with an almost constant quality factor and approximately linear attenuation over a desired frequency band. These properties are very useful for applications in oil prospecting and seismology (Carcione, 2001). As discussed in Liu *et al.* (1976), the distribution of relaxation times represents many different physical relaxation mechanisms. Using this model we make the shear and undrained modulus complex and frequency dependent, while all the other coefficients remain real. Then, the complex moduli  $\widehat{K}_c = \widehat{K}_c(\omega, P_{ef}^b, \bar{S}_n)$  and  $\widehat{N} = \widehat{N}(\omega, P_{ef}^b)$  are computed as

$$\widehat{K}_c = \frac{K_c^r(P_{ef}^b, \bar{S}_n)}{R_{K_c}(\omega, P_{ef}^b) - iT_{K_c}(\omega, P_{ef}^b)} , \quad \widehat{N} = \frac{N^r(P_{ef}^b)}{R_N(\omega, P_{ef}^b) - iT_N(\omega, P_{ef}^b)} . \quad (7.3)$$

The real coefficients  $K_c^r(P_{ef}^b)$  and  $N^r(P_{ef}^b)$  denote the relaxed closed bulk and shear moduli, respectively. They are chosen so that the high frequency limits of Eqs. (7.3) match the values of  $N(P_{ef}^b)$  and  $K_c(P_{ef}^b, \bar{S}_n)$  obtained from Eqs. (7.1) and (2.14), respectively. The frequency dependent functions  $R_l$  and  $T_l$ ,  $l = K_c, N$ , associated with a continuous spectrum of relaxation times, characterize the viscoelastic behaviour and are given by (see Liu *et al.*, 1976; Carcione, 2001)

$$R_l(\omega, P_{ef}^b) = 1 - \frac{1}{\pi \widehat{Q}_l(P_{ef}^b)} \ln \frac{1 + \omega^2 T_{1,l}^2}{1 + \omega^2 T_{2,l}^2} , \quad (7.4)$$

$$T_l(\omega, P_{ef}^b) = \frac{2}{\pi \widehat{Q}_l(P_{ef}^b)} \tan^{-1} \frac{\omega(T_{1,l} - T_{2,l})}{1 + \omega^2 T_{1,l} T_{2,l}} , \quad l = K_c, N. \quad (7.5)$$

The parameters in Eqs. (7.4) and (7.5) are taken such that the resulting quality factors  $Q_l = T_l/R_l$  are approximately equal to the reference values  $\widehat{Q}_l$  in the range of frequencies where the model is applied. To introduce in our model the dependence of the quality factors on effective pressure (Winkler and Nur, 1979), we varied the coefficients  $\widehat{Q}_{K_c}$  and  $\widehat{Q}_N$  from 40 (for the minimum effective pressure) to 60 (for the maximum) and from 48 to 75, respectively. These values were chosen taking into account the fact that for partially saturated rocks the quality factor

for shear deformations is greater than the associated to compressional deformations (Winkler and Nur, 1979). For simplicity any possible dependence of such coefficients on saturation was disregarded. For the computations we take  $T_{1,l} = -\frac{1}{2\pi 10^9}$  s,  $T_{2,l} = -\frac{1}{2\pi 10^9}$  s, for  $l = K_c, N$ .

The value of the Kozeny-Carman constant  $A_0$  in Eqs. (6.6) is equal to 5. This same formulation was recently applied by the authors in Carcione *et al.* (2004), Santos *et al.* (2004) for numerical simulation of wave fronts and satisfies the restrictions imposed by the Laws of Thermodynamics (see the Appendix).

According to Eqs. (6.1) and (6.2) the characteristic angular frequencies at 10 % gas saturation are about  $\omega_c^n = 430$  kHz,  $\omega_c^w = 89$  kHz and  $\omega_c^{nw} = 182$  kHz.

For the following experiments the sample is subjected to a fixed confining pressure of 60 MPa and the pore space is assumed to be filled with water (as the wetting phase) and a hydrocarbon gas. Their properties are:  $\rho_w = 1000$  kg/m<sup>3</sup>,  $\mu_w = 0.01$  N s/m<sup>2</sup>,  $K_w = 2.223$  GPa,  $\rho_n = 100$  kg/m<sup>3</sup>,  $\mu_n = 0.00015$  N s/m<sup>2</sup>,  $K_n = 0.022$  GPa. To obtain plots vs. effective pressure, the reference pressure of the water was varied within the range  $25 \text{ MPa} \leq \bar{P}_w \leq 60 \text{ MPa}$ , keeping constant saturation

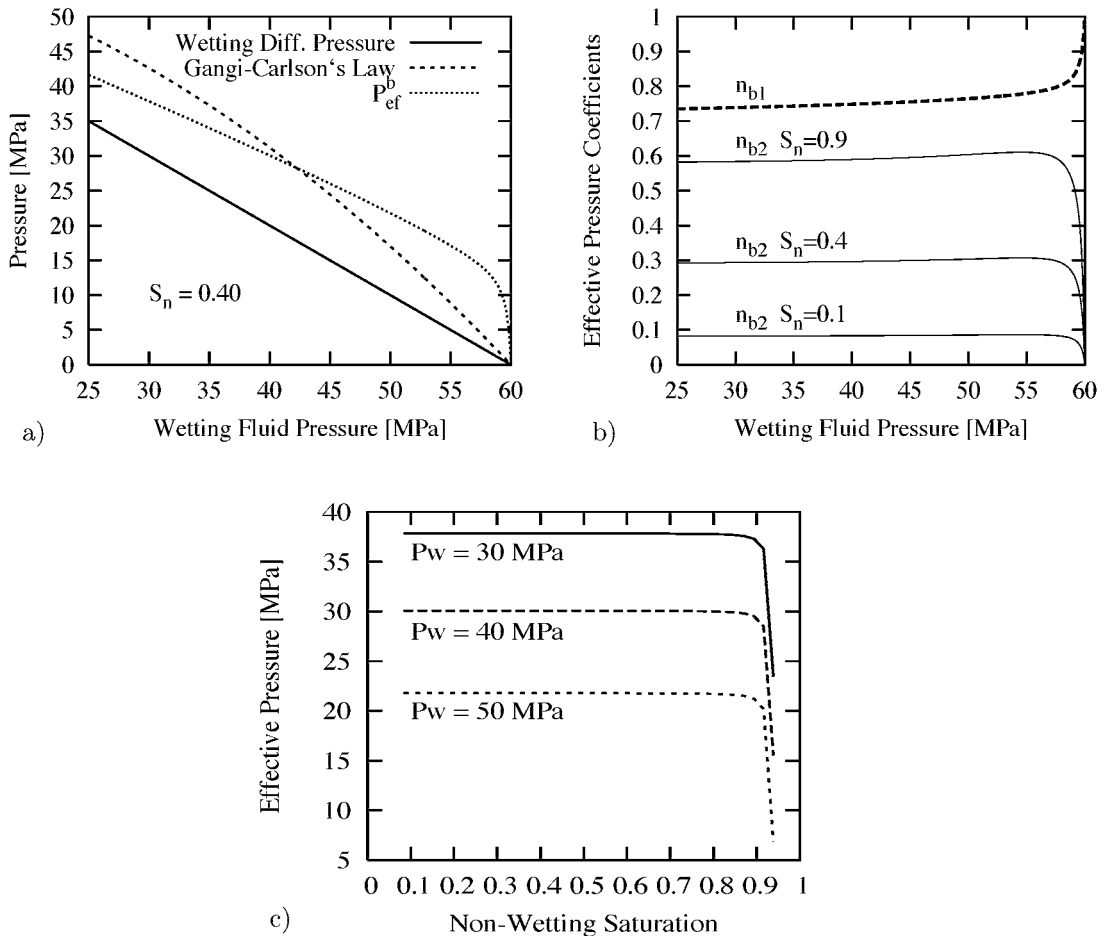


Fig. 1 - a) Comparison between effective pressure, wetting differential pressure and Gangi-Carlson's type law for  $S_n = 0.4$ . b) Behaviour of  $n_{b1}$  and  $n_{b2}$  coefficients versus  $P_w$  for different saturation states. c) Bulk effective pressure vs. non-wetting saturation for different wetting pressure states.



states, while for the curves vs. saturation we take  $0.05 < \bar{S}_n < 0.95$ , maintaining fixed values of  $\bar{P}_w$ . Fig. 1a shows the bulk effective pressure ( $P_{ef}^b$ ) given by Eq. (3.5) versus wetting fluid pressure for 40% gas saturation. We compare our results with other estimates, such as the wetting-differential pressure, defined as  $P_{wd} = P_c \bar{P}_w$  and a general law proposed by Gangi and Carlson (1996) of the form:

$$P_{ef}^G = P_c - n_G \bar{P}_w, \quad \text{with } n_G = n_0 - n_1 \bar{P}_{wd}, \quad n_0 = 1, \quad n_1 = 0.014 \text{ MPa}^{-1}.$$

Significant differences can be observed in almost all the pressure range. As expected, the increase in wetting fluid pressure causes a reduction in the effective pressure acting on the solid matrix, producing a “softening” effect in the sandstone. Fig. 1b illustrates the coefficients  $n_{b1}$  and  $n_{b2}$  given in Eqs. (3.6) and (3.7) versus  $\bar{P}_w$  for different saturation states. We observe that when  $\bar{P}_w$  approaches  $P_c$  (i.e., near the fracture limit),  $n_{b1} \rightarrow 1$  and  $n_{b2} \rightarrow 0$ , and  $P_{ef}^b \rightarrow P_{wd}$  (since  $\bar{C}_{bc} \rightarrow \infty$  for  $\bar{P}_w \rightarrow P_c$ ). Unlike the “classic”  $n_{b1}$  coefficient,  $n_{b2}$  is strongly dependent on saturation. To analyze the influence of the capillary forces on the bulk effective pressure, in Fig. 1c we plot  $P_{ef}^b$  vs. non-wetting saturation for different values of  $\bar{P}_w$ . It can be observed that the curves are almost constant with saturation, except near the irreducible water saturation, where they show an abrupt decrease. This behaviour should be carefully investigated with appropriate laboratory measurements.

Next we analyze the behaviour of the phase velocities and quality factors [using Eqs. (6.8)] for the different wave propagation modes in the ultrasonic frequency range (for  $f = 1$  MHz). In Fig. 2a we plot the phase velocity of the Type I compressional wave vs.  $P_{ef}^b$  for 10, 40 and 90% gas saturation. According to Eqs. (7.2) the capillary pressures at these saturation states are 3.226 KPa, 9.859 KPa and 1.2 MPa, respectively. The marked increase observed in the velocity with effective pressure reflects the change in the elastic properties of the skeleton with effective pressure given by Eq. (7.1), and is mainly associated with the closure of microcracks, low aspect ratio pores and loose grain contacts, which increase the stiffness of the rock. For very low effective pressures this velocity shows a significant decrease, an effect usually observed in

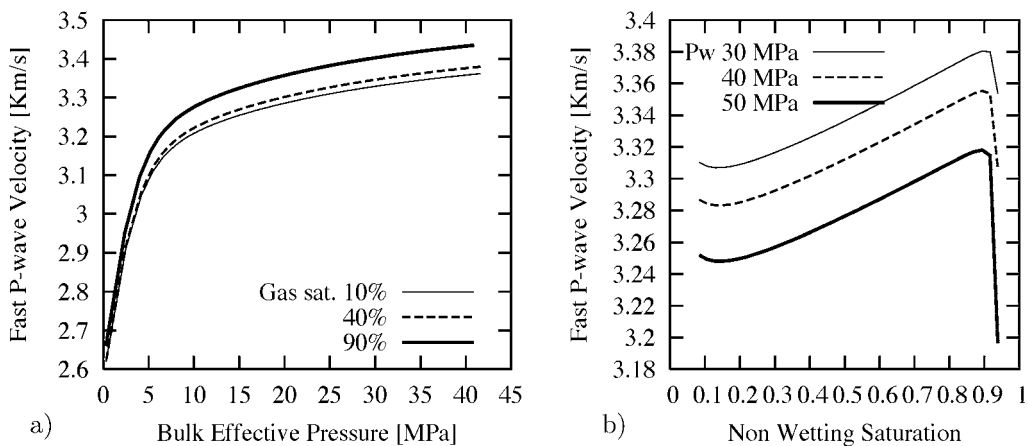


Fig. 2 - Phase velocity of Type I - P waves for a frequency  $f = 1$  MHz: a) vs.  $P_{ef}^b$  for some fixed saturations and b) vs. non-wetting saturation  $\bar{S}_n$  for different fixed wetting pressures.

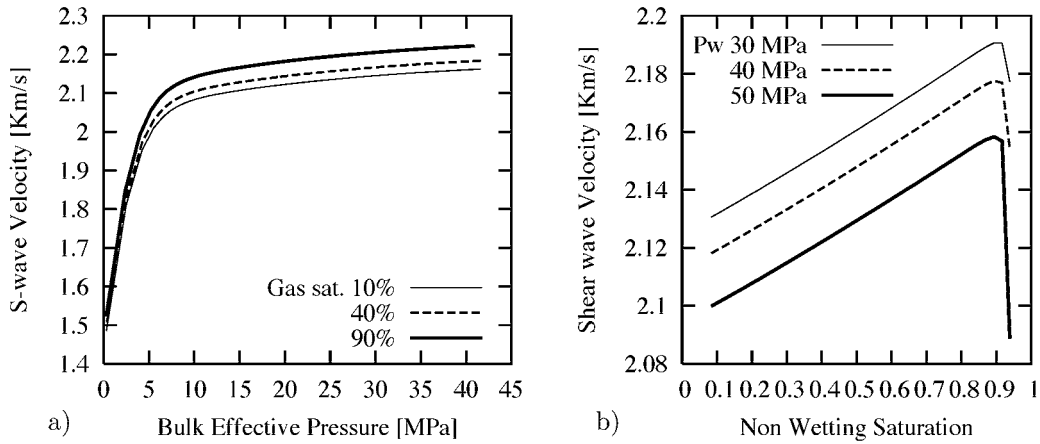


Fig. 3 - Phase velocity of shear waves for a frequency  $f = 1\text{MHz}$ : a) vs.  $P_{ef}^b$  for some fixed saturations and b) vs. non-wetting saturation  $\bar{S}_n$  for different fixed wetting pressures.

formations with very high pore pressures (i.e. overpressured), and also predicted by different theories (see Toksöz *et al.*, 1976; Carcione and Gangi, 2000).

At these pressure conditions the velocity is not sensitive to saturation, a behaviour that was also checked with an isostress fluid mixture model (i.e. a classical Biot model). Similar results were presented by Toksöz *et al.* (1976) for this sandstone (see Fig. 12 in that paper). We also study the dependence of the Type I - P wave velocity on non-wetting saturation for fixed wetting pressures  $\bar{P}_w = 30\text{ MPa}$ ,  $40\text{ MPa}$  and  $50\text{ MPa}$  and this is shown in Fig. 2b. The effective pressures for these experiments are about 37, 30 and 22 MPa, respectively. As can be observed this velocity reflects the combined effects of the variations in the poroviscoelastic moduli, the effective pressure law and in the bulk density with saturation. The absolute minimum corresponds to the

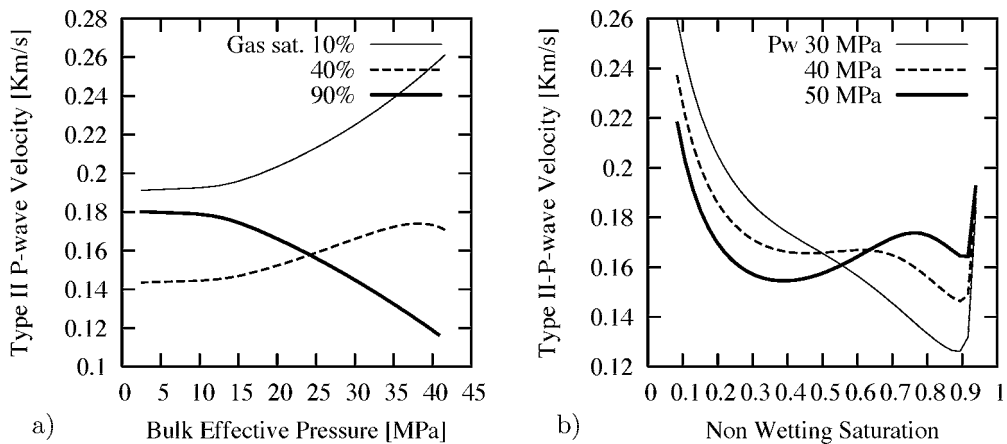


Fig. 4 - Phase velocities of Type II - P waves for a frequency  $f = 1\text{MHz}$ : a) vs.  $P_{ef}^b$  for some fixed saturations and b) vs. non-wetting saturation  $\bar{S}_n$  for different fixed wetting pressures.

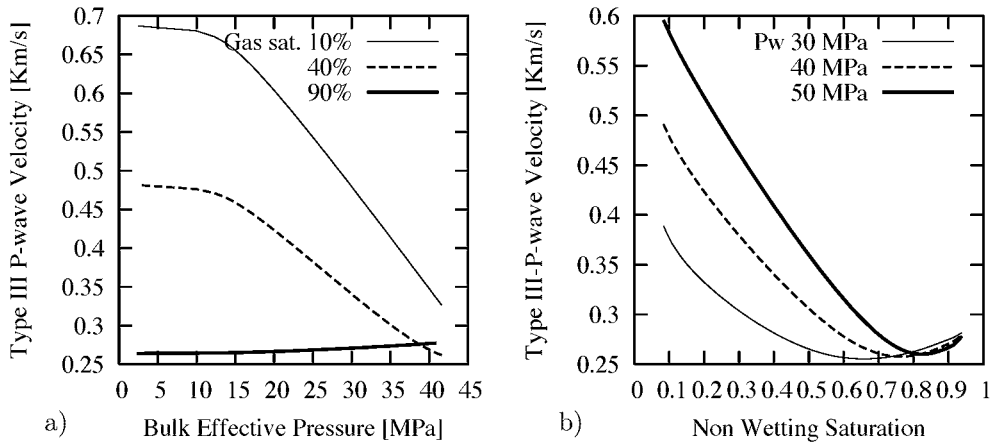


Fig. 5 - Phase velocities of Type III - P waves for a frequency  $f = 1\text{MHz}$ : a) vs.  $P_{ef}^b$  for some fixed saturations and b) vs. non-wetting saturation  $\bar{S}_n$  for different fixed wetting pressures.

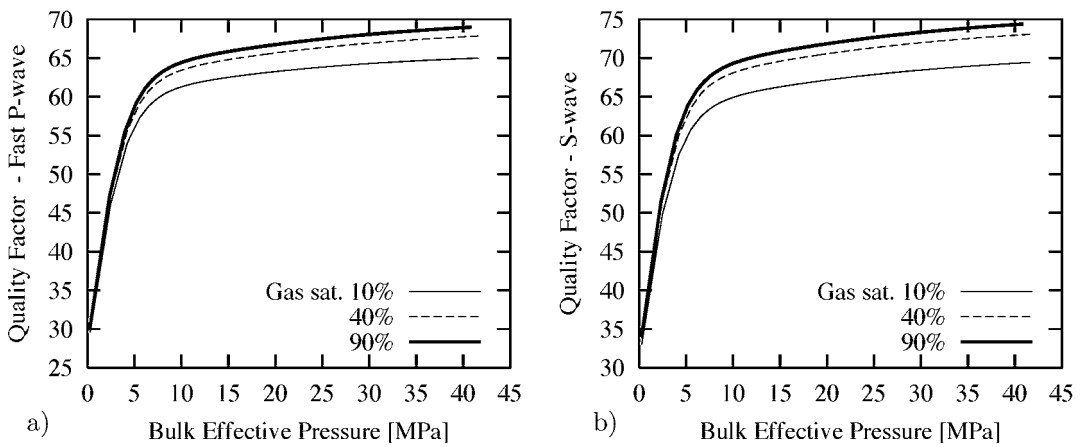


Fig. 6 - Quality Factors of Type I - P and shear waves vs.  $P_{ef}^b$  for different saturations.

effect observed in the effective pressure near the residual water saturation. A quite similar behaviour is observed in Fig. 3, where we plot the phase velocity of shear waves vs. effective pressure and vs. saturation. Next we analyze the dependence of phase velocities of the slow waves with saturation and effective pressure. From Figs. 4 and 5 we observe that Type II and Type III-P wave velocities are clearly more sensitive to saturation also showing a strong dependence on effective pressure.

The quality factors for the Type I -P and shear waves vs.  $P_{ef}^b$ , are shown in Fig. 6. As observed in the phase velocities, these coefficients show an important monotonic increase with effective pressure, in agreement with the observations made by different authors (see Tao *et al.*, 1995; Schön, 1996; Carcione and Gangi, 2000). The quality factors associated to the slow waves are shown in Fig. 7. Except at intermediate saturations, the  $Q_{II}$  coefficient shows only slight variations with pressure changes, being more sensitive to saturation. The type III quality factor  $Q_{III}$  shows a stronger dependence on these variables.

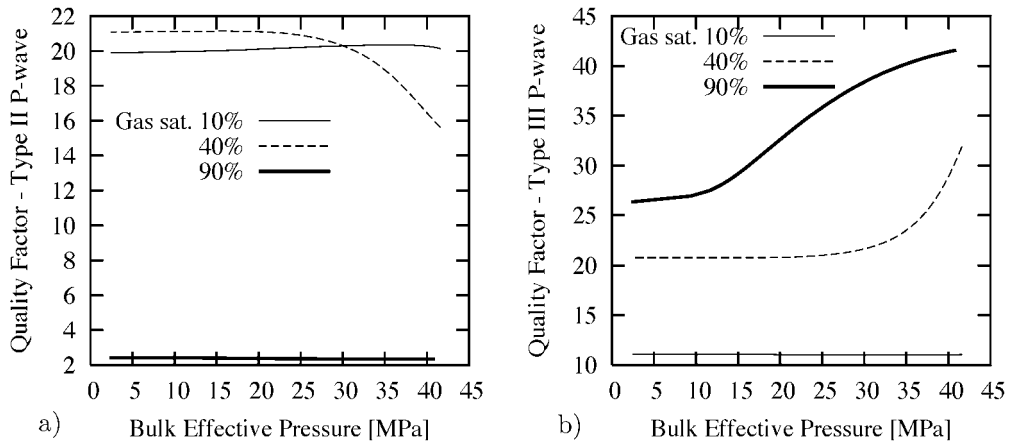


Fig. 7 - Quality Factors of Type II and III - P and shear waves vs.  $P_{ef}^b$  for different saturations.

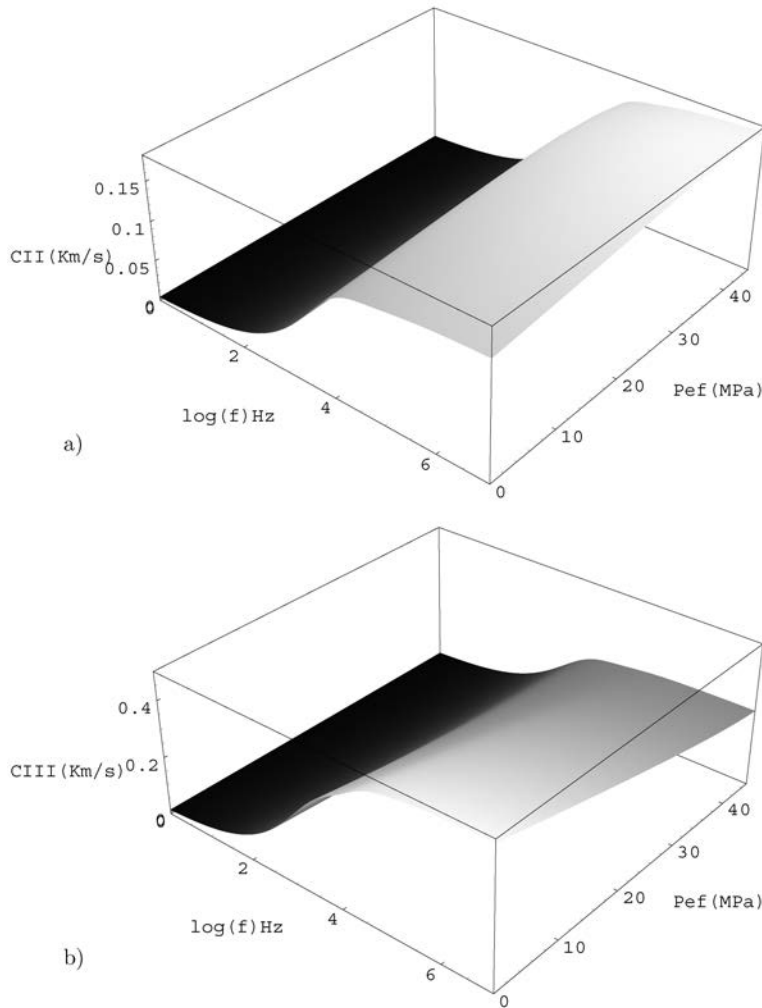


Fig. 8 - Behaviour of phase velocities of Type II and Type III- P waves versus effective pressure and vs. linear frequency  $f$  (in log. scale) for 40% gas saturation ( $S_n = 0.40$ .)

It is also interesting to analyze the frequency dependence of the slow P - wave velocities and quality factors under different effective pressures. This is shown in the surface plots corresponding to Figs. 8 and 9, where the linear frequency  $f$  varies from 1 Hz to  $10^7$  Hz (in logarithmic scale) and  $S_n = 0.4$ . It is observed that these slow compressional modes have very low velocities and quality factors in the low frequency range, with a marked increase with frequency. This indicates that these waves are observable only at very high frequencies (like those used for ultrasonic testing of rock samples in laboratory). This is also in agreement with previous results (Santos *et al.*, 1990b), indicating that the dispersive character of the slow waves is not really influenced by the pressure conditions of the reservoir. The third slow compressional wave was numerically simulated in some recent works by Carcione *et al.* (2004), Santos *et al.* (2004).

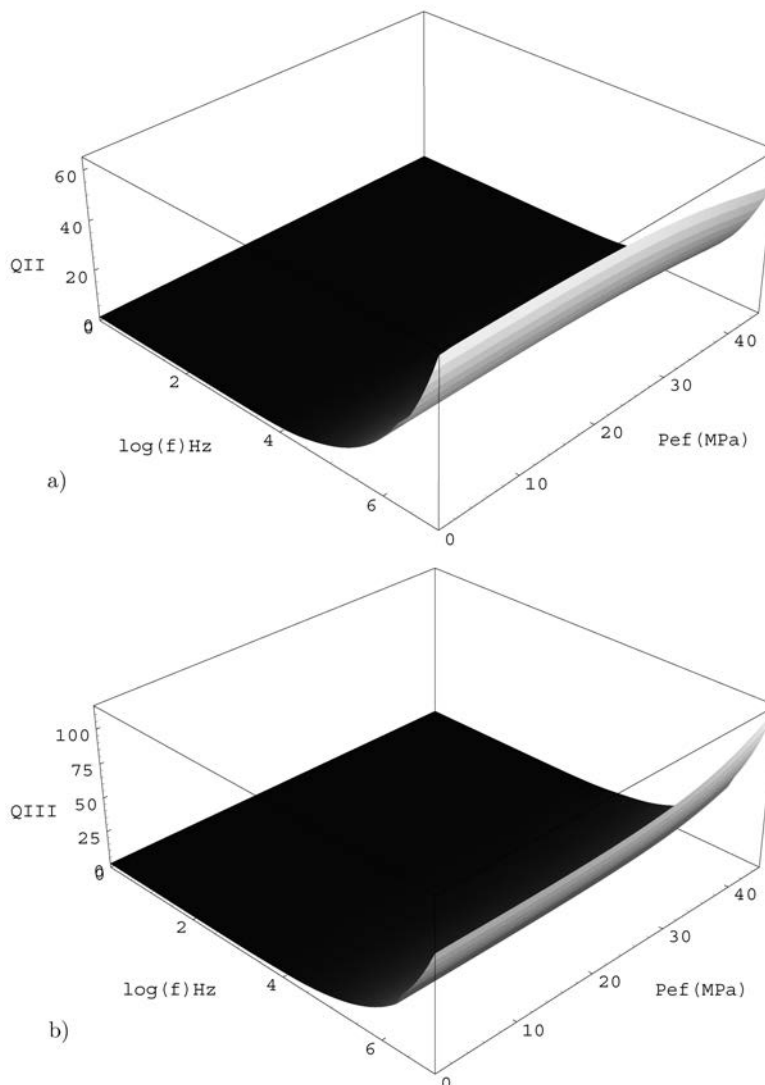


Fig. 9 - Behaviour of quality factors of Type II and Type III- P waves versus effective pressure vs. linear frequency  $f$  (in logscale) for 40% gas saturation ( $S_n = 0.40$ .)

## 8. Conclusions

In this work we developed a theory to study the processes of deformation and wave propagation in porous solids saturated by two immiscible fluids. The model allows the inclusion of many parameters such as porosity, permeability, lithology, pore fluid types, saturation state, confining pressure, fluid pressures and the capillary pressure vs. saturation curve, which are very important in different fields such as rock physics, interpretation of laboratory data and reservoir characterization.

We derived the elastic stress-strain relations, introducing appropriate elastic moduli. Generalizing a classic approach we defined a set of compressibilities and established a new effective pressure law for elastic bulk volume deformations. Its behaviour is consistent with other estimates and models the well known effect of decreasing of seismic velocities observed in overpressured formations. Although the  $\bar{n}_{b2}$  coefficient is strongly dependent on the saturation state, except near the irreducible water saturation, the capillary forces do not produce important changes on effective pressure. The procedure can be readily extended to analyze pore volume deformations.

The equations of motion, valid from the seismic to the ultrasonic range, were formulated in the space-frequency domain, taking into account attenuation and dispersion effects associated to frequency dependent viscodynamic effects and viscoelasticity. Using the principles of continuum thermodynamics we generalized some inequalities valid for single-phase viscoelastic media and established restrictions to the poroviscoelastic moduli.

The model was applied to study the influence of saturation and variable fluid pressures on dilatational and shear wave velocities and the corresponding quality factors in a sample of Boise sandstone saturated by water and gas. The wave velocities are very sensitive to effective pressure and in particular, those of the Type II and Type III waves are strongly dependent on saturation state. This is also observed in the corresponding quality factors. This suggests that combined analysis of phase velocities and quality factors can be used as indicators of the saturation and pressure states of a reservoir rock. The results relative to Type I and shear waves are in good agreement with published experimental and theoretical works. Those corresponding to the slow waves should be checked in the laboratory and we hope this will motivate further experimental work in this subject.

The different results presented in this work demonstrate that in transition zones within hydrocarbon reservoirs, where in general two (or more) immiscible fluids saturate the pore space, a three-phase model allows for a more accurate description and interpretation of its acoustic response. In particular, a quantitative analysis of the amount of energy converted from fast P and S waves to slow waves at the heterogeneities present in this kind of media would be important and will be the subject of future publications.

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## Appendix. Thermodynamic restrictions

Here we derive a set of restrictions imposed by the Laws of Thermodynamics on the imaginary parts of the coefficients in Eqs. (5.1). It will be assumed that the principles of continuum thermodynamics are valid for our system at the macroscopic scale. A different approach was

presented by De la Cruz *et al.* (1993), based on the energy balance equations in the solid and the fluid phases and using averaging techniques. Combining the first and second laws of thermodynamics and using a generalization of the argument given in Fabrizio and Morro (1992) for viscoelastic materials and in Ravazzoli (1995) for poroviscoelastic solids saturated by single-phase fluids, for any isothermic cycle of period  $2\pi/\omega$  the following inequality must hold:

$$\int_0^{2\pi/\omega} (\Delta\tau_{ij}\dot{\varepsilon}_{ij} + \Delta\mathcal{T}_n\dot{\xi}^n + \Delta\mathcal{T}_w\dot{\xi}^w) dt \geq 0 \tag{A1}$$

Next, we choose  $\varepsilon_{ij}$ ,  $\xi^n$  and  $\xi^w$  in Eq. (A1) and in the stress-strain relations (5.2) to be of the form  $\varepsilon_{ij} = A_{ij}(\cos\omega t + \sin\omega t)$ ,  $\xi^l = \xi^l(\cos\omega t + \sin\omega t)$ ,  $\omega \geq 0$ , where  $A_{ij}$  is a symmetric matrix. Set  $e_A = A_{ii}$  and note that  $e_b(t) \equiv \varepsilon_{ii} = e_A(\cos\omega t + \sin\omega t)$ . A lengthy calculation yields

$$e_A^2 \dot{R}_{\lambda c}^S(\omega) + \dot{R}_N^S(\omega)A_{ij}A_{ij} - 2\xi^n e_A \dot{R}_{B_1}^S(\omega) - 2\xi^w e_A \dot{R}_{B_2}^S(\omega) + (\xi^n)^2 \dot{R}_{M_1}^S(\omega) + (\xi^w)^2 \dot{R}_{M_2}^S(\omega) + 2\xi^n \xi^w \dot{R}_{M_3}^S(\omega) \leq 0, \quad \omega \geq 0, \tag{A2}$$

which is an extension of Graffi's inequality for this type of media (see Fabrizio and Morro, 1992).

Setting  $\widehat{K}_{cl}(\omega) = \widehat{\lambda}_{cl}(\omega) + \frac{2}{3}\widehat{N}_l(\omega)$  and combining Eq. (5.3) and the identity

$$2A_{ij}A_{ij} = \frac{2}{3} [e_A^2 + (A_{11} - A_{22})^2 + (A_{11} - A_{33})^2 + (A_{22} - A_{33})^2] + 4(A_{12}^2 + A_{13}^2 + A_{23}^2),$$

we see that Eq. (A2) can be stated in the form

$$e_A^2 \widehat{K}_{cl}(\omega) - 2\xi^n e_A \widehat{B}_{1l}(\omega) - 2\xi^w e_A \widehat{B}_{2l}(\omega) + (\xi^n)^2 \widehat{M}_{1l}(\omega) + (\xi^w)^2 \widehat{M}_{2l}(\omega) + 2\xi^n \xi^w \widehat{M}_{3l}(\omega) + 4[(A_{12}^2 + A_{13}^2 + A_{23}^2) + \frac{2}{3}[(A_{11} - A_{22})^2 + (A_{11} - A_{33})^2 + (A_{22} - A_{33})^2]] \widehat{N}_l(\omega) \geq 0, \quad \omega \geq 0. \tag{A3}$$

Next we consider the case of a hydrostatic compression, i.e.,  $A_{ij} = 0$ ,  $i \neq j$  and  $A_{11} = A_{22} = A_{33}$ . From Eq. (A3) we get

$$\vec{Z}^t \widehat{P}_l(\omega) \vec{Z}^t \geq 0, \quad \omega \geq 0, \tag{A4}$$

where  $\vec{Z}^t = (e_A, \xi^n, \xi^w)$  and the symmetric matrix  $\widehat{P}_l$  is defined by  $\widehat{P}_{111} = \widehat{K}_{cl}$ ,  $\widehat{P}_{122} = \widehat{M}_{1l}$ ,  $\widehat{P}_{133} = \widehat{M}_{2l}$ ,  $\widehat{P}_{112} = -\widehat{B}_{1l}$ ,  $\widehat{P}_{113} = -\widehat{B}_{2l}$ ,  $\widehat{P}_{123} = \widehat{M}_{3l}$ .

Next for a pure shear oscillation, i.e.,  $A_{ij} \neq 0$ ,  $i \neq j$  and  $A_{11} = A_{22} = A_{33} = 0$  and  $\xi_1^n = \xi_1^w = 0$ , from Eq. (A3) we have

$$\widehat{N}_l(\omega) \geq 0, \quad \omega \geq 0. \tag{A5}$$

Inequalities (A4) and (A5) are the thermodynamic restrictions imposed on the imaginary parts of the complex frequency dependent coefficients in the stress-strain relations (5.1).

**REMARK.** The equality in Eqs. (A4) and (A5) holds if and only if  $\omega = 0$ , i.e., the system behaves as elastic at zero frequency. This results from Eq. (5.3) and the definitions of the half-range Fourier sine and cosine transforms.

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