Potential field determination due to the given masses of a Digital Terrain Model for a system of target points

D. TSOULIS

Berlin University of Technology, Berlin, Germany

(Received, August 8, 2002; accepted May 29, 2003)

Abstract - For the computation of the gravitational field of a given mass distribution, as defined through the sampled heights of a Digital Terrain Model (DTM), several analytical as well as numerical methods exist. The computation may take place at individual points/gravity stations, or over a system of target points, which can either coincide with the DTM-points or may lay on the horizontal plane at a certain flight height above the terrain. The possibility of evaluating the gravitational terrain effect simultaneously for all points of a regular grid, made numerical methods, in particular those based on the fast Fourier Transform (FFT), a very popular tool in geodesy. The present paper offers an up-to-date survey on different analytical and numerical methods for the evaluation of terrain effects using the information of a densely sampled DTM of a test area situated in the Bavarian Alps. It is demonstrated, that the choice of the modeling technique for the surface relief can be made only with analytical methods. The choice between an analytical or a numerical method in the practical application should be made keeping in mind the respective accuracy of the final results, were the analytical methods are shown to be clearly superior.

1. Introduction

The computation of the gravitational effect of given mass distributions is a central research issue in disciplines such as applied geophysics, physics and geodesy. In the mid-Sixties numerous articles on the gravity field modeling of ideal bodies appeared, originating mainly from the community of exploration geophysicists who were involved from a practical viewpoint in relevant exploration problems. An ideal body model describes a density distribution of a

Corresponding author: D. Tsoulis, Berlin University of Technology, Sekr H12, Strasse des 17. Juni 135, 10623 Berlin, Germany; phone: +49 3031424054, fax: +49 3031421973; e-mail: tsoulis@mca.bv.tu-berlin.de

© *2003 OGS*

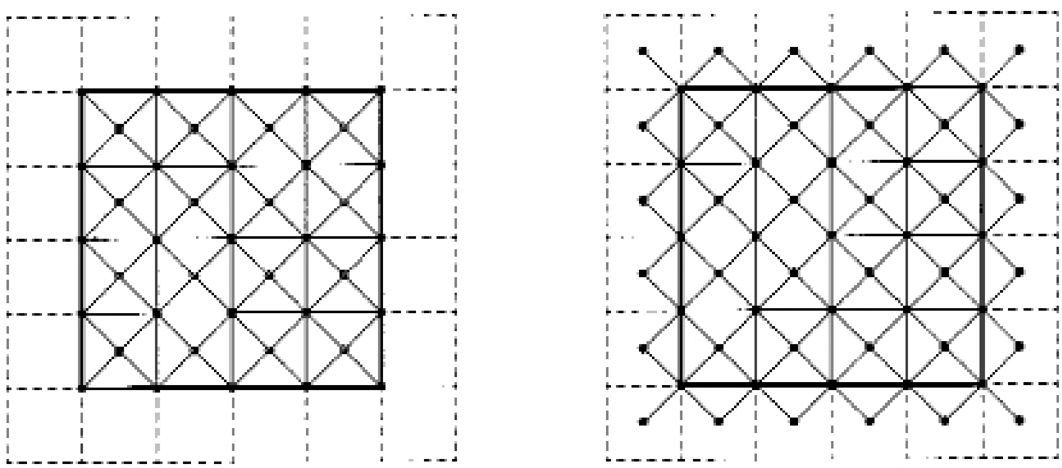
completely known geometrical shape. The advantage of such a modeling method lies in the fact that a mass distribution of a prism, a cylinder, a cone or a horizontal plate produces a gravity field that can be described exactly, i.e. by means of closed analytical expressions which are some mixture of transcendental functions of the relative position of the computation point with respect to the known source. In some cases, the derivation of such closed expressions for the gravitational potential, and its derivatives up to the second order, can become very tedious, as in the case of a target point situated outside the symmetry axis of a circular cylinder. Still, analytical formulae for the gravity field of the most common model bodies are available, see e.g. Mader (1951), Jung (1961), Nagy (1966), Barnett (1976), Chen and Cook (1993) and consult Tsoulis (1999) for an exhaustive literature. The precise evaluation of the gravitational effect of known mass distributions became again very popular in the last decades thanks to recent research on the so-called fifth force or some new theories in fundamental physics such as gravitational waves. Furthermore, analytical and numerical methods for the computation of the satellite's gravitational signal and its temporal variations for points inside it, using model bodies to approximate the spacecraft, have been considered in simulation studies of ongoing and planned satellite missions for the study of the Earth's gravity field (see e.g. Lockerbie et al., 1993; Müller, 2001).

While the complexity of the modeling method is of minor importance when a limited number of target points is used, it can become a rather serious problem if a raster of computation points is of interest, which is the case in local and regional geoid determination or airborne gravimetry and gradiometry. The computation of the so-called terrain corrections can become a very time consuming process in such applications, when one applies the aforementioned analytical methods. The introduction of spectral techniques for the computation of convolution integrals describing quantities of the gravity field in the early Seventies offered an efficient alternative to the analytical computation of the respective quantities. The main feature of these methods is that they deliver the desired quantity very quickly for all points of a grid, simultaneously, thus obtaining a prominent position among the analytical or numerical techniques available up to that point for the solution of the same problems (Parker, 1972; Dorman and Lewis, 1974; Sideris, 1984; Schwarz et al., 1990). Although the respective series may be divergent for a computation point situated on the topography, the Fast Fourier Transform (FFT) methods established themselves rapidly in physical geodesy and are currently used as one of the most popular techniques among geodesists and geophysicists worldwide. For the computation of terrain effects, in particular, a combination method has been recently proposed which merges the efficiency of the FFT methods with the rigor of the analytical methods removing at the same time the divergence of the FFT-series when densely sampled Digital Terrain Models are used (Tsoulis, 1999, 2001).

The aim of the present paper is to compare some of the most frequently used methods - including two proposed very recently - for the computation of terrain corrections in terms of their efficiency as well as accuracy. For this purpose a densely sampled DTM (15 km × 20 km, $dx = dy = 50$ m) of a part of the Bavarian Alps is used. Among the purely analytical, the purely numerical and the combined solutions there exists a relation between the efficiency of the applied method and the envisaged accuracy of the result, which this paper is trying to establish.

2. Representation of the surface relief

The term terrain, as found in geophysics and geodesy and complying to our common sense is used to describe a piece of the Earth's surface over continents. Mathematically speaking, the terrain is a 2-dimensional surface in 3-dimensional space which can be regarded as the graph of a function $f : A \subset \Re^2 \rightarrow \Re$ that assigns a height $f(q)$ to every point q in the domain A of the terrain. The domain A is unfortunately never known in an absolute sense. What we do have at our disposal, is the knowledge of the height with respect to some reference surface at selected points of the Earth's surface, derived usually through photogrammetric analysis of the respective stereo-photographs. From the value of the function f at this finite set $Q \subset A$ of sample points we have to somehow approximate the height at the other points of the domain. A first approach for approximating the terrain, which leads to a more or less natural image of the unknown reality, is a triangulation of Q, i.e. a definition of bounded faces of A as the triangles which emerge by connecting the sample points of Q or of another set obtained by some interpolation of the original known data of Q, e.g. by building over four neighboring sample points (see Fig. 1). In any case, these triangles represent the faces and the sample points of the vertices of Q. Thus, we approximate the domain Q through a piecewise linear continuous function. It should be stressed, however, that we are referring to one of many approximations to one of many subsets of the actual terrain domain A. For example, the processing of another stereo-pair over the same region will provide another set of sample points Q'. Nevertheless, we use the derived *polyhedral terrain* as an approximation of the actual topographic relief. However, it is obvious that this approach, for the representation of a terrain using the discrete information of a DTM, leaves much to be desired. Since we do not know how the actual topographic surface looks like we can never be sure what kind of triangulation is the most appropriate for our purpose. As a matter



(a) (b)

Fig. 1 - Polyhedral triangulation on the basis of a DTM for a grid of mean values over four neighboring sample points, using the original height data as (a) point values or (b) block mean values.

of fact, any kind of triangulation using a certain set of sample points does the job equally well, provided that the numerical values for the heights at the given points are treated as correct. In the case of randomly sampled points, one can deduce a so-called optimal triangulation from a finite number of possible triangulations by selecting the one which maximizes the minimum angle of the respective triangular faces (de Berg et al., 2000). Unfortunately, such a criterion cannot be applied to the regularly gridded data of a DTM, see Fig. 1. Here, the choice of the segment dividing the tetrahedron connecting four neighboring sample points into two triangular faces can be made in both x or y directions resulting in a finite number of different yet justified approximation models for the unknown terrain. One of the many possible triangulations is shown in Fig. 1.

3. Polyhedral versus bilinear modeling

Of special interest in the study of analytical solutions for the gravitational field of ideal bodies is the case of the general polyhedron. We consider a polyhedral source of constant density consisting of an arbitrary number of faces, each of which is defined by a variable number of segments and/or vertices. The polyhedron gravitation was first derived in the mid-1800s. Contemporary papers include, among others, Barnett (1976), Götze (1976), Pohanka (1988), Strakhov and Lapina (1990), Werner (1994), Petrović (1996), Werner and Scheeres (1997), Tsoulis (1999), Tsoulis and Petrović (2001). When the data of a DTM are used, then the problem of the triangulation method for the representation of the surface relief is reduced to a choice between the models presented in Fig. 1 or a similar meshing scheme. The construction of the closed surface including the masses of a DTM is a straightforward task. The problem can become tedious, however, when the only information about the source available are the coordinates of the vertices defining its boundary in some reference system. The absence of extra information about convex or concave structures in terms of additional constraints can turn a simple exercise to a non-solvable problem in computational geometry (de Berg et al., 2000). First results on the application of the polyhedron modeling for the computation of terrain corrections over the points of a DTM are presented in Skiba (1999) and Tsoulis (1999, 2001).

A refinement of the polyhedron modeling was proposed by Tsoulis et al. (2001). The four triangles defined for each computation point by the four neighboring DTM-points, are replaced by a bilinear surface going through the data points. This more delicate - from the modeling point of view - approach results in a one dimensional integration of a function with - apparently - no analytical solution. The performance of the respective numerical integration has to be evaluated taking into account the theory of adaptive integration methods (Forsythe et al., 1977).

Fig. 2 displays the two test points located on the surface of the densely sampled Bavarian DTM, for which a comparison between the polyhedron and the bilinear modeling for the computation of the terrain effect due to the known masses defined by the Digital Terrain Model is attempted. Tables 1 and 2 compare the computation of the terrain effect due to the masses implied by the DTM of Fig. 2 using both the polyhedron and the bilinear modeling

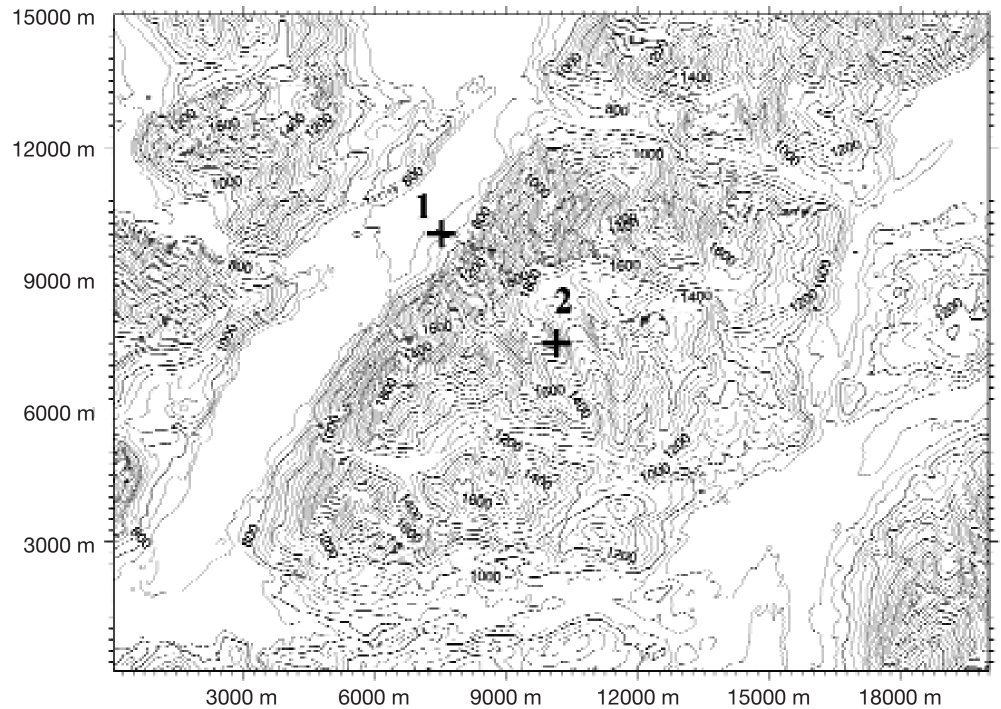


Fig. 2 - A comparison between the polyhedron and the bilinear modeling at two individual target points located on the surface of the used 15 km x 20 km DTM of the Bavarian Alps.

methods. The calculations take place at the two target points marked in Fig. 2 and the adaptivity investigation for the numerical integration arising in the bilinear formulae is realized by splitting the bilinear calculations into two zones: an intermediate or inner zone, where the number of the integration intervals is let to increase until a level of absolute accuracy in the numerical integration is obtained, and an outer zone where the number of the subintervals is

Table 1 - Comparison between polyhedron and bilinear modeling of the surface relief in terms of the number of subintervals in the resulting one-dimensional integration. Test point 1.

Number of subintervals in the near zone	**stretching of the near zone**							
	50 m		**100 m**		**250 m**		**500 m**	
	U	U_z	U	U_z	U	U_z	U	U_z
	$[m^2/s^2]$	$[mGal]$	$[m^2/s^2]$	$[mGal]$	$[m^2/s^2]$	$[mGal]$	$[m^2/s^2]$	$[mGal]$
	ΔU	ΔU_z	ΔU	ΔU_z	ΔU	ΔU_z	ΔU	ΔU_z
	$[\sim\mu m]$	$[\mu Gal]$	$[\sim\mu m]$	$[\mu Gal]$	$[\sim\mu m]$	$[\mu Gal]$	$[\sim\mu m]$	$[\mu Gal]$
2	10.513	-60.079	10.513	-60.310	10.513	-60.309	10.513	-60.309
	-0.298	230.327	-0.003	- 0.729	-0.002	- 0.186	-0.002	- 0.015
4	10.513	-60.185	10.513	-60.309	10.513	-60.309	10.513	-60.309
	-0.165	124.458	-0.002	- 0.068	-0.002	- 0.012	-0.002	- 0.001
8	10.513	-60.349	10.513	-60.309	10.513	-60.309	10.513	-60.309
	0.052	-39.378	-0.000	- 0.004	0.000	- 0.001	-0.000	0.000
16	10.513	-60.306	10.513	-60.309	10.513	-60.309	10.513	-60.309
	-0.004	2.920	0.000	0.000	0.000	0.000	0.000	0.000

Table 2 - Comparison between polyhedron and bilinear modeling of the surface relief in terms of the number of subintervals in the resulting one-dimensional integration. Test point 2.

Number of subintervals in the near zone	stretching of the near zone							
	50 m		100 m		250 m		500 m	
	U	U_z	U	U_z	U	U_z	U	U_z
	$[m^2/s^2]$	$[mGal]$	$[m^2/s^2]$	$[mGal]$	$[m^2/s^2]$	$[mGal]$	$[m^2/s^2]$	$[mGal]$
	ΔU	ΔU_z	ΔU	ΔU_z	ΔU	ΔU_z	ΔU	ΔU_z
	$[\sim\mu m]$	$[\mu Gal]$	$[\sim\mu m]$	$[\mu Gal]$	$[\sim\mu m]$	$[\mu Gal]$	$[\sim\mu m]$	$[\mu Gal]$
2	11.191	-163.857	11.191	-163.872	11.191	-163.871	11.191	-163.871
	0.175	14.700	-0.001	- 1.102	0.002	- 0.140	0.001	- 0.010
4	11.191	-163.871	11.191	-163.871	11.191	-163.871	11.191	-163.871
	0.003	0.011	0.001	- 0.054	0.001	- 0.009	0.001	- 0.001
8	11.191	-163.871	11.191	-163.871	11.191	-163.871	11.191	-163.871
	0.001	0.001	0.001	- 0.004	0.001	- 0.001	0.001	- 0.000
16	11.191	-163.871	11.191	-163.871	11.191	-163.871	11.191	-163.871
	0.000	0.001	0.000	- 0.000	0.000	- 0.000	0.000	- 0.000

allowed to vary arbitrarily. The term absolute accuracy is used here to describe the floating point accuracy defined on the respective platform (for our computations the value $eps = 1 \cdot 10^{-19}$ is applied). The number of integration intervals is increased until the difference between two successive integrations is less or equal to *eps*. In this respect, the critical inner zone can be seen as an area where an exact evaluation of the one-dimensional integration is carried out, while the choice of the number for the subintervals correlates with the size of the inner zone. Tables 1 and 2 present some test computations of the bilinear method for the two computation points displayed in Fig. 2. The results obtained were compared to the respective quantities by applying the polyhedral modeling according to Fig. 1b. The two quantities considered are the gravitational potential and the vertical component of the gravitational attraction. The latter was computed for the bilinear method in mGal, the differences with the polyhedral modeling are expressed in µGal, to intensify the comparison. The potential is evaluated for the bilinear method in m^2 / s^2. The differences in potential between the two methods are expressed in approximated differences in geoid height through the formula of Bruns ($N = T / \gamma$) and are given in µm. The information given in these tables leads to the conclusion that for the inner zone of 100 m and a choice of two integration intervals for the rest of the height data the bilinear and the polyhedron methods are practically identical with the difference between the two being at the sub-µGal to µGal level.

4. Terrain effects over a grid of target points

Using the data of the Bavarian DTM displayed in Fig. 2 we now compute the terrain correction quantity not for individual points but for a regular grid of target points coinciding with the DTM data, i.e. the computation points lie on the surface of the topographic relief and are identical to the corresponding sample points of the DTM (compare Fig. 1). The different methods demand a different amount of CPU time to carry out the respective computations.

The analytical methods, for example, whose evaluation principle is to sum up the contribution of discrete finite elements modeled by ideal bodies such as prisms, cylinders etc. in order to deliver the gravity effect at an individual target point, demand a CPU time of $\beta \cdot N \cdot M$ seconds, where N is the total number of data points, M the number of computation points and β a proportionality constant, depending on the complexity of the used model and of course on the speed of the available computer (a point mass modeling requires a repeated evaluation of the simple inverse square distance function, whereas the flat top or rectangular prism method involves, for a single data point, the computation of a rather complicated trigonometric function, a total of eight times). A much faster alternative is offered by the so-called spectral methods, in particular the FFT techniques. These numerical methods were applied in geodesy, mainly, to assist computations carried out over a regular grid of data points. When the data points coincide with the target points, as in local and regional geoid applications and local gravity field modeling, the FFT approach gives the desired gravity field quantity (in the context of the present discussion the terrain correction) at all points of the grid, simultaneously, claiming a CPU time of $\alpha \cdot N \cdot \log N$ seconds with α being again, the respective proportionality constant varying between different computational platforms (Sideris, 1984). The FFT methods, consuming a much smaller amount of CPU time than for example the prism summation method, are often described as an efficient method. It has to be stressed here, however, that the term efficiency should be always used in combination with the accuracy obtained by the respective method. The linear approximation of the terrain correction (1st term of the FFT series) offers for example a representation of the terrain by means of mass lines, neglecting the actual dimensions of the finite prismatic compartments. It is obvious, that the final numerical quantity that is produced thereby, can represent, only very poorly, the actual gravitational effect of the topographical masses.

The computational effort in terms of CPU time, for both bilinear and polyhedral modeling, is comparable to that of the traditional prism summation method (Forsberg, 1984). On a Pentium 3 personal computer running at 1 GHz, which corresponds to a rather medium performance by today's standards, bilinear and polyhedron methods consumed almost three days of computer time to deliver the terrain correction quantity at all 120000 grid points of the 15 km × 20 km Bavarian DTM of resolution $dx = dy = 50$ m. For a comparison of the results, we need a reference computation, which can be made only arbitrarily, as no modeling method can claim to be more advanced, in a direct comparison, to another one. All methods attempt to approximate differently the same unknown reality. The polyhedral modeling, which approximates the variations of the relief much better than the step function representation of the rectangular prism method, is chosen as the reference computation for the assessment of the different terrain correction techniques applied herein.

Table 3 compares the traditional prism summation method, the bilinear method, the combination method proposed by Tsoulis (1999) (polyhedron triangulation up to 1.5 km away from each computation point, the contribution of the rest of the DTM masses through a modified FFT series up to the third order) and the so-called linear approximation of the terrain correction quantity (the first term of an unmodified FFT series) with the polyhedral modeling (Fig. 1b). In order to enhance the presentation, the signs of the differences are kept in the search of minima

Table 3 - A comparison between different terrain correction computation methods with the polyhedral modeling method. Differences expressed in mGal.

	MIN	MAX	MEAN	SD
Prisms - Polyhedra	- 8.6477	6.6061	1.1853	1.3025
(Bilinear modeling) - Polyhedra	- 0.1127	0.0234	0.0061	0.0029
(Combination method) - Polyhedra	- 8.4698	8.4615	1.0730	1.2984
(Linear Approximation) - Polyhedra	-11.0088	6.9050	2.0775	1.8058

and maxima and in the computation of standard deviations, whereas the absolute values of these differences are considered when evaluating their mean values. First of all, it becomes evident that the bilinear and polyhedral methods are almost identical. This was expected since the choice of 100 m for the inner zone and a number of two integration intervals for the rest of the area supplied an agreement for the same DTM between the two methods at the μGal level (Tsoulis et al., 2001). What is also clear, are the substantial discrepancies between polyhedra and the rest of the methods. The differences vary between a couple and 11 mGal, depending on the roughness of the terrain surrounding the computation point. The hybrid solution proposed by Tsoulis (1999) is comparable to the rectangular prism summation method, as is indicated by the similar statistics. The use of flat prisms for the contribution of the outer zone should be the reason why the combination technique is almost identical to the prism method. Finally, the linear approximation to the terrain correction quantity yields the worst results compared to the polyhedra. The mass line approximation of each elementary DTM field causes discrepancies bigger than the flat prism method.

5. Discussion and conclusions

The existence of heterogeneous modeling and evaluation methods for the computation of topographical effects in gravity field modeling applications deserves a closer look and a detailed assessment of the different approaches. It is clear that the contribution of the neighborhood of the computation points is very critical and an extra effort should be spent for the exact modeling of the masses in this near zone, even when the computation takes place for a system of target points. Combination methods proved applicable for large data sets, since they combine the accuracy of the analytical with the speed of the numerical methods. The ever faster and ever better processors, now conquering the market, begin to attenuate the argument against analytical methods which are very time consuming, and consolidating modeling methods, such as polyhedra or bilinear surfaces, as feasible alternatives for routine applications. The non negligible discrepancies between the different methods reported here should encourage researchers to emphasize the issue of accuracy. Although, the real terrain remains unknown, there are some modeling methods that approximate it better than others. Thus, the efficiency of one method should not only concentrate on the speed of the respective computations, but should also keep in mind its accuracy in terms of the model used to represent the Earth's surface. Polyhedral modeling proved to be identical with the use of bilinear surfaces for the approximation of the surface relief, with respect to the gravity quantities computed for the same grid of data. In the future, each of these methods may be equally considered as the reference method in important relevant computations.

References

Barnett C.T.; 1976: *Theoretical modeling of the magnetic and gravitational fields of an arbitrary shaped three-dimensional body*. Geophysics, **41**, 1353-1364.

Chen Y.T. and Cook A.H.; 1993: *Gravitational Experiments in the Laboratory*. Cambridge University Press, Cambridge, 268 pp.

Dorman L.M. and Lewis B.T.R.; 1974: *The use of nonlinear functional expansions in calculation of the terrain effect in airborne and marine gravimetry and gradiometry*. Geophysics, **39**, 33-38.

de Berg M., van Kreveld M., Overmars M. and Schwarzkopf O.; 2000: *Computational Geometry*. Springer Verlag, Berlin, 367 pp.

Forsberg R.; 1984: *Study of terrain reductions, density anomalies and geophysical inversion methods in gravity field modelling*. Reports of the Department of Geodetic Science and Surveying, Report No. 355, Ohio State University.

Forsythe G.E., Malcolm M.A. and Moler C.B.; 1977: *Computer methods for mathematical computations*. Prentice-Hall Inc., New Jersey, 259 pp.

Götze H.J.; 1976: *Ein numerisches Verfahren zur Berechnung der gravimetrischen und magnetischen Feldgrössen für drei-dimensionale Modellkörper*. Dissertation, Technische Universität Clausthal.

Jung K.; 1961: *Schwerkraftverfahren in der Angewandten Geophysik*. Akademische Verlagsgesellschaft Geest und Portig, Leipzig, 348 pp.

Lockerbie N.A., Veryaskin A.V. and Xu X.; 1993: *Differential gravitational coupling between cylindrically-symmetric, concentric test masses and an arbitrary gravitational source: relevance to the STEP experiment*. Class. Quantum Grav., **10**, 2419-2430.

Mader K.; 1951: *Das Newtonsche Raumpotential prismatischer Körper und seine Ableitungen bis zur dritten Ordnung*. Öster Zeitschr Vermess, Sonderheft 11, 74 pp.

Müller J.; 2001: *Die Satellitengradiometriemission GOCE, Theorie, technische Realisierung und wissenschaftliche Nutzung*. Deutsche Geodätische Kommission, Reihe C., Heft Nr 541, München, 87 pp.

Nagy D.; 1966: *The gravitational attraction of a right rectangular prism*. Geophysics, **31**, 362-371.

Parker R.L.; 1972: *The rapid calculation of potential anomalies*. Geoph. J. R. Astr. Soc., **31**, 447-455.

Petrović S.; 1996: *Determination of the potential of homogeneous polyhedral bodies using line integrals*. J. Geod., **71**, 44-52.

Pohánka V.; 1988: *Optimum expression for computation of the gravity field of a homogeneous polyhedral body.* Geoph. Prosp., **36**, 733-751.

Schwarz K.P., Sideris M.G. and Forsberg R.; 1990: *The use of FFT techniques in physical geodesy*. Geoph. J. Int. **100**, 485-514.

Sideris M.G.; 1984: *Computation of gravimetric terrain corrections using Fast Fourier Transform techniques*. University of Calgary, Department of Civil Engineering, Division of Surveying Engineering, Report No. 20007, 110 pp.

Skiba P.; 1999: *Zur Anwendung von Polyedern und Quadern bei der Berechnung des Schwerefeldes topographischer Massen*. Diplomarbeit, Technische Universität Berlin, Institut für Geodäsie und Geoinformationstechnik, 47 pp.

Strakhov V.N. and Lapina M.I.; 1990: *Direct gravimetric and magnetometric problems for homogeneous polyhedrons*. Geophys. J. (UK), **8**, 740-756.

Tsoulis D.; 1999: *Analytical and numerical methods in gravity field modelling of ideal and real masses*. Deutsche Geodätische Kommission, Reihe C, Heft Nr 510, München, 133 pp.

Tsoulis D.; 2001: *Terrain correction computations for a densely sampled DTM in the Bavarian Alps*. J. Geod., **75**, 291-307.

Tsoulis D. and Petrović S.; 2001: *On the singularities of the gravity field of a homogeneous polyhedral body.* Geophysics, **66**, 535-539.

Tsoulis D., Wziontek H. and Petrović S.; 2001: *The use of bilinear surfaces in the computation of gravitational effects due to topographic masses given on a regular grid*. In: Proc. of Analytical Representation of Potential Field Anomalies for Europe, (AROPA Workshop), Institute d'Europe, Castle of Münsbach, Luxembourg, 23-27.10.2001, in press.

Werner R.A.; 1994: *The gravitational potential of a homogeneous polyhedron, or, Don't cut corners*. Celest. Mechan. Dyn. Astr., **59**, 253-278.

Werner R.A. and Scheeres D.J.; 1997: *Exterior gravitation of a polyhedron derived and compared with harmonic and mascon gravitation representations of asteroid 4769 Castalia*. Celest. Mechan. Dyn. Astr., **65**, 313-344.