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# Superstatistics, complexity and earthquakes: a brief review and application on Hellenic seismicity

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***ABSTRACT*** A short overview on superstatistics applied in various complex systems is firstly provided by roughly outlining the methodology, including the deduction of superstatistical parameters from time series and the characterisation of the complex system as a special type of superstatistics. Applications of superstatistics on various physical systems are reported and originally performed on Greek seismicity. The application of the superstatistical methodology, investigating the dynamics of seismogenesis in Greece, revealed the complex turbulent nature of the earthquake data. The verification of the superstatistical hypothesis strongly depends on the parameter used for differencing. In the case the hypothesis is verified the results evidence log-normal superstatistics, otherwise the results indicate either Gaussian distributed random variables of constant variance with outliers or Tsallis $q_{stat}$-Gaussian variables with constant variance.

**Key words:** superstatistics, complex systems, time series analysis, seismicity, Greece.

## 1. Introduction

### *1.1. Theory*

Superstatistics is a concept developed for describing the statistics of complex systems far from equilibrium, in which the stationary states are characterised on long time scales by large fluctuations of their intensive quantities (e.g. inverse temperature, friction constant, chemical potential or energy dissipation). Variations of the statistical properties of these fluctuations lead to different effective statistical mechanical descriptions (Beck and Cohen, 2003). The simplest example of a superstatistical system is the motion of a Brownian particle of mass $m$ through a $d$-dimensional inhomogeneous medium in which the temperature varies both in space and time. In this example, the particle wanders in a certain cell with a given temperature for a very short period, then moves to the next cell with a different temperature and so on (Beck and Cohen, 2003). This indicates that there are two dynamics: one fast corresponding to the velocity of the Brownian particle and a slow one related to the temperature variations of the medium (Beck, 2006). Now, the particle velocity  can be efficiently described by a linear Langevin equation:

$$\frac{d\vec{v}}{dt} = \gamma F(\vec{v}) + \sigma \vec{W}_t \tag{1}$$

where $\vec{W}(t)$ corresponds to Gaussian white noise component, $\gamma > 0$ is a friction constant, $\sigma$ determines the strength of noise component and $F(\vec{v}) = -(\partial/\partial\,\vec{v})\,E\,(\vec{v})$ is a drift force (Beck, 2004).

 

Each spatial cell is characterised by an intensive parameter $\beta$ (here this parameter coincides with inverse temperature) given by $\beta = 2\gamma / (m\sigma^2)$. If the medium is homogeneous, then the parameters $\gamma$ and $\sigma$ are constant and so is the intensive parameter $\beta$, resulting an ordinary Brownian motion. Nonetheless, in a non-equilibrium situation, $\gamma$ and $\sigma$ are parameters slowly fluctuating on a much higher time scale than velocity $\vec{v}$ (Beck, 2009). Hence, $\beta$ also fluctuates from one cell to another getting a probability density $f(\beta)$. Locally, for a time scale $T$ and for each cell, $\beta$ can be considered approximately constant and then the local stationary distribution in each cell is Gaussian, given by:

$$p(\vec{v} \mid \beta) = \left(\frac{\beta}{2\pi}\right)^{d/2} e^{-\frac{1}{2}\beta E} \tag{2}$$

where $E = \frac{1}{2} m\vec{v}^2$ is the kinetic energy of the particle. However, in the long term behaviour of the particle, namely for times $t >> T$, the marginal distribution $p(\vec{v})$, namely the probability of observing a certain value of $\vec{v}$ irrespective of what $\beta$ is (Beck, 2004), is given by:

$$p(\vec{v}) = \int_0^\infty p(\vec{v} \mid \beta) f(\beta) d\beta \tag{3}$$

As it will be described analytically in Section 2, depending on the behaviour of $f(\beta)$ generalisations of different superstatistical models are possible such as $\chi^2$, inverse $\chi^2$, log-normal, $F$-superstatistics among others (Beck, 2006).

Similar to this example and keeping the same mathematical formalism, one may consider a more general case in which a macroscopic system, under non-equilibrium steady states, consists of many smaller cells which reach local equilibrium very fast (e.g. relaxation time $\tau$ for each cell is very small). In this system there exists an intensive parameter $\beta$ exhibiting spatio-temporal fluctuations, while locally, i.e. in spatial regions (cells), is approximately constant for a time scale $T$. In addition, depending on the physical problem, the variable $\vec{v}$ can be appropriately replaced with another variable describing the problem considered. For example, for modelling turbulence in superstatistical framework, $\vec{v}$ can be replaced with the velocity difference $\vec{u}$ or acceleration $\vec{a}$ on smallest scales (Beck *et al.*, 2005).

In each cell, the system is described by ordinary statistical mechanics, i.e. ordinary Boltzmann factors $e^{-\beta E}$, where $E$ is an effective energy in each cell (in the previous example $E$ was the kinetic energy). However, in the long term ($t >> T$), the system is described by a spatio-temporal average over the fluctuating $\beta$ and in this way one obtains a superposition of two statistics, namely that of $\beta$ and that of $e^{-\beta E}$. This is the base for the term of "superstatistics" (Beck and Cohen, 2003; Beck, 2004, 2006; Beck *et al.*, 2005) for which the stationary distribution can be given as a superposition of Boltzmann factors $e^{-\beta E}$ weighted by the probability density $f(\beta)$ to detect a value $\beta$ in a random cell, namely:

$$p(E) = \int_0^\infty \frac{1}{Z(\beta)} f(\beta)\rho(E)e^{-\beta E} d\beta \tag{4}$$

where $\rho(E)$ is the density of states and $Z(\beta)$ is the normalisation constant of $\rho(E)\, e^{-\beta E}$ for a given $\beta$ (Beck *et al.*, 2005). In Fig. 1, inspired and modified from Beck (2011), a macroscopic system is

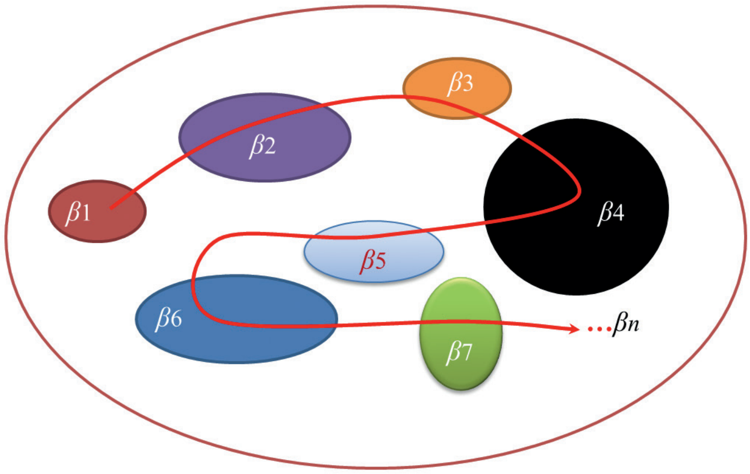


Fig. 1 - An example of spatially inhomogeneous macroscopic system where the ellipses with different colors represent different mesoscopic systems (cells) embedded into a fluctuation environment with different intensive parameters $\beta$ [inspired and modified from Beck (2011)].

depicted where the ellipses with different colours represent different mesoscopic systems (cells) embedded into a fluctuation environment with different intensive parameters $\beta$.

### *1.2. Applications*

Superstatistics has been applied to describe efficiently various physical phenomena. For example, the measured densities of velocity differences in the fully developed hydrodynamic turbulence are in excellent agreement with log-normal superstatistics, whereas the velocity differences of defect turbulence (a phenomenon related to convection) coincide quite well with typical predictions of Tsallis non-extensive models (Beck, 2004). In the same study, a paradigm from high energy physics is described concerning high-energy collision processes induced by astrophysical sources, which lead to the creation of cosmic ray particles verifying the non-extensive behaviour of the measured cosmic rays. Atmospheric turbulence was also studied (Rizzo and Rapisarda, 2005) applying superstatistics to a temporal series of turbulent wind measurements and reproducing very well the fluctuations and the probability density functions of wind velocity returns and differences. Superstatistical analysis of train delay data was performed (Beck, 2008) revealing their Tsallis $q$-exponential distributions. Summarising, other applications include spectral fluctuations of billiards (Abul-Magd *et al.*, 2008), financial dynamics (Van der Straeten and Beck, 2009; Denys *et al.*, 2016; Xu and Beck, 2016), cancer survival statistics (Leon and Beck, 2008), currents in complex polymers (Yalcin and Beck, 2012), temperatures (Yalcin and Beck, 2013), sea level fluctuations (Rabassa and Beck, 2015), atmospheric Hg0 concentration data series from different latitudes (Carbone *et al.*, 2018), among others. In all these studies, at least one of the four superstatistical classes was verified, indicating the superstatistical nature of the overall processes.

Earthquake generation processes are very complex, characterised by intermittency and nonstationary clustering (Orlecka-Sikora *et al.*, 2019). Even though results derived from the superstatistics theory are not yet reported for describing seismic process complexity, at least based on authors' knowledge, a connection exists since in many studies the concept of Tsallis nonextensive statistical mechanics (NESM) (Tsallis, 1988) was effectively applied. As it will be shown in next paragraphs, these results can be related to the sub-class of superstatistics, namely $\chi^2$-superstatistics (or Gamma distribution), as presented in Section 2.1. In particular, in a series of papers (this is a non-exhaustive list), a thorough investigation of seismicity behaviour, known to exhibit fractality and long-range interactions, took place based on NESM, which was proved as a very efficient approach for the statistical description of the nonlinear dynamics and evolution of various seismogenic systems, such as Greek region (Vilar *et al.*, 2007; Sarlis *et al.*, 2010; Telesca, 2010, 2012; Iliopoulos *et al.*, 2012; Vallianatos *et al.*, 2012, 2013, 2014a, 2014b, 2016, 2018; Michas *et al.*, 2013; Papadakis *et al.*, 2013, 2016; Antonopoulos *et al.*, 2014; Pavlos *et al.*, 2014, 2018; Efstathiou *et al.*, 2017; Papadakis and Vallianatos, 2017; Chochlaki *et al.*, 2018; Michas and Vallianatos, 2018; and references therein). Various models were also developed based on NESM, such as the fragment asperity model (e.g. Sotolongo-Costa and Posadas, 2004; Silva *et al.*, 2006) in which a mechanism of earthquake triggering is derived by the combination of the roughness of the fault planes and the fragment-size distribution. Overall, the NESM seems to provide an efficient framework for the description of seismicity, with significant input towards understanding the way that this extreme complexity is manifested in multi-spatiotemporal scales. A special mention must be given to a Bayesian model which, through Mathai's pathway model, can be related to superstatistics, the application of which to Chilean strong earthquakes provided promising results (Sánchez and Vega-Jorguera, 2018).

The interest in superstatistics is substantiated from the above retrospection, since for many physical systems this concept yields a plausible explanation for non-trivial, non-Gaussian distributions $p(E)$ (e.g. Tsallis distributions) occurrence (Beck, 2004). Within its framework, the observed non-Gaussian, non-extensive behaviour arises naturally due to the spatio-temporal fluctuations of $\beta$, while locally the behaviour is described by ordinary statistical mechanics.

## 2. Types of superstatistics

As mentioned above, different examples of possible superstatistics can be generated depending on different realisations of distributions $f(\beta)$, which cannot be Gaussian since $\beta$ is always positive by construction (Beck *et al.*, 2005).

### *2.1. $\chi^2$-superstatistics*

If the probability density of $\beta$ is given by the $\chi^2$-distribution (or equivalently the Gamma distribution):

$$f(\beta)=\frac{1}{\Gamma\left(\frac{n}{2}\right)}\left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}}\beta^{\frac{n}{2}-1}\exp\left\{-\frac{n\beta}{2\beta_0}\right\} \tag{5}$$

then the distribution resulting from Eqs. 4 and 5 exhibits power-law tails for large $|\vec{v}|$, leading to Tsallis statistics (Beck *et al.*, 2005). Considering the simple dynamic realisation paradigm described in introduction with linear drift forces namely $F(\vec{v}) = -\vec{v}$ and $\beta=\gamma/\sigma^2$, the marginal probability of an observed variable $\vec{v}$ is given by:

$$p(\vec{v}) \sim \frac{1}{\left(1+\frac{\beta_0}{n}\vec{v}^2\right)^{\frac{n+1}{2}}} \tag{6}$$

where $n$ is the number of degrees of freedom, $\beta_0$ is the average of $\beta$ given by $\beta_0 = \int_0^\infty \beta f(\beta)d\beta$ and the variance $\sigma^2 = \frac{2}{n}\beta_0^2$. The degrees of freedom, $n$, under appropriate modifications (Beck, 2004) can be related to a parameter $q$ with $q = 1+2/n$ which is very close to Tsallis entropic index given by $q_{stat} = 1 + 2/(n+1)$ (Beck *et al.*, 2005), and, therefore, Eq. 6 yields the generalised canonical distributions of nonextensive statistical mechanics (Beck, 2009).

### *2.2. Inverse $\chi^2$-superstatics*

If we consider $\beta^{-1}$ instead of $\beta$ the resulting $f(\beta)$ is the inverse $\chi^2$-distribution given by:

$$f(\beta)=\frac{\beta_0}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{n\beta_0}{2}\right)^{n/2}\beta^{-n/2-2}e^{-n\beta_0/(2\beta)} \tag{7}$$

which generates superstatistical distributions that present exponential decays (Beck *et al.*, 2005).

### *2.3. Log-normal superstatistics*

If $f(\beta)$ follows the log-normal distribution:

$$f(\beta)=\frac{1}{\beta s\sqrt{2\pi}}\exp\left\{\frac{-\left(\log\frac{\beta}{m}\right)^2}{2s^2}\right\} \tag{8}$$

where $m$ and $s$ are parameters, the average of $\beta$ is given by $\beta_0 = m\sqrt{w}$ and the variance $\sigma^2 = m^2w(w-1)$ where $w := e^{s^2}$. Again, considering the paradigm of the moving particle for linear drift forces, the generalised canonical distribution of $\vec{v}$ is given by the superstatistical distribution:

$$p(v) = \frac{1}{2\pi s}\int_0^\infty d\beta\beta^{-1/2}\exp\left\{\frac{-\left(\log\frac{\beta}{m}\right)^2}{2s^2}\right\}e^{-\frac{1}{2}\beta v^2} \tag{9}$$

Even though the integral in Eq. 9 cannot be evaluated in closed form, it is easily numerically estimated.

### *2.4. F-superstatistics*

Another example concerns superstatistics based on $F$-distributions given by:

$$f(\beta)=\frac{\Gamma((a+w)/2)}{\Gamma(\alpha/2)\Gamma(w/2)}\left(\frac{ba}{w}\right)^{\alpha/2}\frac{\beta^{\frac{a}{2}-1}}{(1+\frac{ab}{w}\beta)^{(\alpha+w)/2}} \tag{10}$$

where $a$ and $w$ are positive integers and $b > 0$ is a parameter. For $a = 2$ a Tsallis distribution is obtained in $\beta$-space and not in $\upsilon$-space. In this case, the average of $\beta$ is given by $\beta_0 = \frac{w}{b(w-2)}$ and the variance $\sigma^2 = \frac{2w^2(a+w-2)}{b^2a(w-2)^2(w-4)}$. Similarly to log-normal distribution for $\beta$, the integral in Eq. 10 can be easily numerically estimated, even though a closed form cannot be obtained.

## 3. Methodology

Given some experimental time series $u(t)$, the hypothesis of superstatistics is tested (Rizzo and Rapisarda, 2005) by extracting the two basic scales $\tau$ and $T$, as well as the PDF $f(\beta)$ (Beck *et al.*, 2005). For the realisation of this test, $u(t)$ is sliced in small pieces in which the time series becomes almost Gaussian. The superstatistical approach is also given schematically in Fig. 2, inspired and modified from Laubrich *et al.* (2008). As already mentioned $u(t)$ could be any time series measured for a physical system, but it can be also a differenced time series (or incremented series) such as $u(\mathrm{t}) = \upsilon(t+\delta)-\upsilon(t)$ of a time series $\upsilon(t)$, where $\delta$ is a time difference, $\delta = 2^j$ with $j = 0, 1, 2, \ldots$ (Beck *et al.*, 2005).

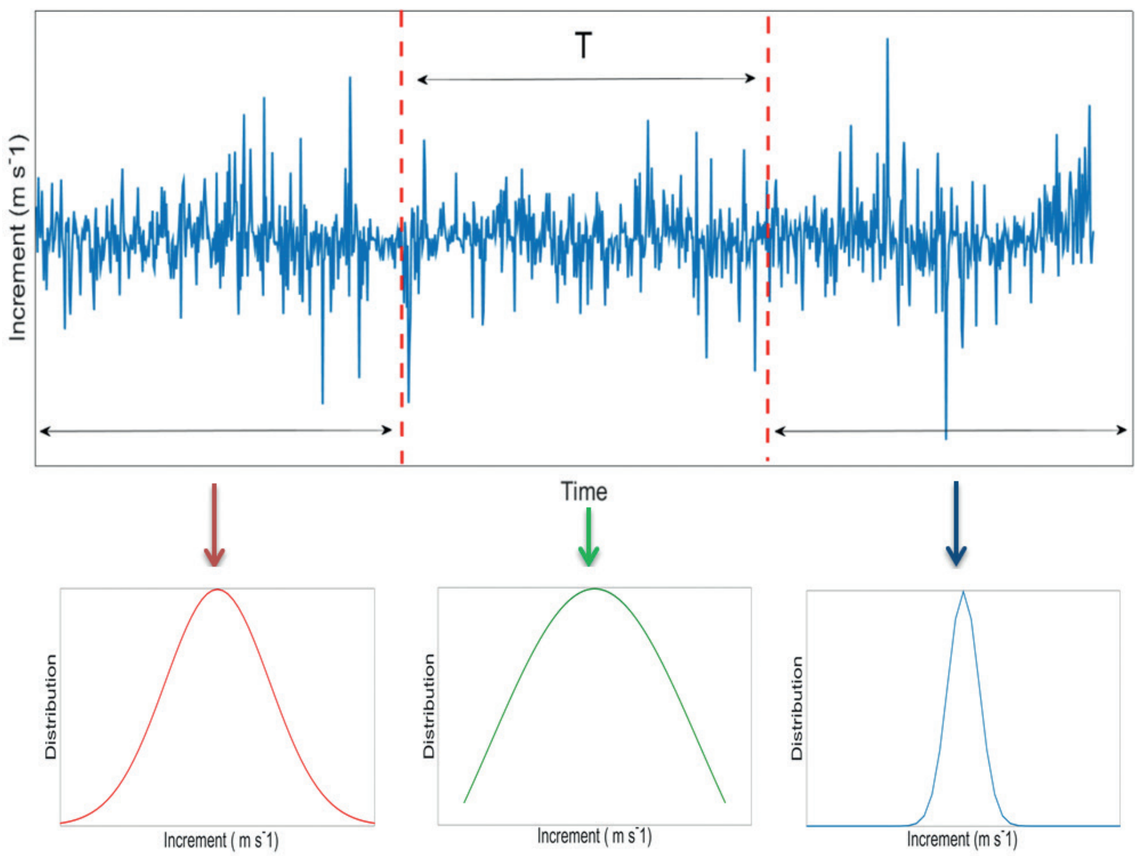


Fig. 2 - Schematic sketch of the superstatistical approach. The increment (differenced) series is considered to consist of successive normally distributed segments of length $T$ [inspired and modified from Laubrich *et al.* (2008)].

### *3.1. Large time scale T*

For the estimation of the large time scale $T$, the time series is firstly divided into $N$ equal time intervals of size $\Delta t$. Therefore, the total length of the signal $u(t)$ is $t_{max} = N\Delta t$, and a function $\varkappa(\Delta t)$ (e.g. kurtosis function) is then defined as:

$$\kappa(\Delta t)=\frac{1}{t_{\max}-\Delta t}\int_{0}^{t_{\max}-\Delta t} dt_0 \frac{\left\langle (u-\bar{u})^4\right\rangle_{t_0,\Delta t}}{\left\langle (u-\bar{u})^2\right\rangle^2_{t_0,\Delta t}} \tag{11}$$

where $\bar{u}$ is the average of $u(t)$ and $\langle ...\rangle_{t_0,\Delta t} = \int_{t_0}^{t_0+\Delta t} ...dt$ stands for an integration over an interval of length $\Delta t$ starting at $t_0$ (Beck *et al.*, 2005). Practically, Eq. 11 implies that local flatness is estimated for each window of length $\Delta t$ and then the result is averaged over all $t_0$'s. Finally, the large time scale $T$ is given by:

$$\varkappa(T) = 3 \tag{12}$$

since $\varkappa(\Delta t) = 3$ for a Gaussian process. Deviation from 3 implies intermittency and non-Gaussian behaviour and, in particular, bigger/smaller values imply super/sub-Gaussian distributions respectively, and intermittency (Iliopoulos and Aifantis, 2018). For very small values of window $\Delta t$, particularly as short as only one measurement of $u$ can be contained in each window, then $\varkappa(\Delta t) = 1$. On the other hand, for large values of window $\Delta t$, particularly as high as that the window $\Delta t$ almost equals to the length of the time series, $\varkappa(\Delta t) > 3$, as expected from superstatistical distributions that are heavy-tailed. This practically means that since the function $k(\Delta t)$ is lower than 3 for small values of $\Delta t$ and higher than 3 for large values, there should be a $\Delta t = T$ for which $\varkappa(T) = 3$. Hence, for each time difference $\delta$ the relevant large time scale $T$ can be extracted.

### *3.2. Short time scale τ*

The short time scale $\tau$ corresponds to the relaxation time of the dynamics and can be evaluated from the short-time exponential decay of the (auto)-correlation functions of the time series $u(t)$. In particular, we are looking for $r_u(k) = e^{-k/\tau}$ (Abul-Magd *et al.*, 2008). When autocorrelation functions do not follow this trend, the empirical value can be parameterised in the form of a superposition of two exponentially decaying functions, as:

$$r_u(k) = A_1 e^{-k/\tau_1} + A_2 e^{-k/\tau_2} \tag{13}$$

where $A_1$=1.5 and $A_2$=-0.5 (Abul-Magd *et al.*, 2008), and $\tau$ is estimated as the mean value of $\tau_1$ and $\tau_2$. Finally, the short time scales $\tau$ are compared with the large time scales $T$ and the ratio $T/\tau$ is evaluated. If the ratio is large compared to unity, then it is sufficient to claim that two well-separated time scales exist justifying the superstatistics' hypothesis.

### *3.3. Estimation of the parameter distribution f(β)*

The slowly varying stochastic variable $\beta$(t) is determined from the time series $u(t)$ as:

$$\beta(t_0) = \frac{1}{\left\langle u^2\right\rangle_{t_0,T} - \left\langle u\right\rangle^2_{t_0,T}} \; . \tag{14}$$

A necessary condition of superstatistics is that this variable, $\beta$, should fluctuate more smoothly (e.g. slow changes) compared to $u(t)$, namely characterised by a long memory and slow correlation decay. Then, the probability density $f(\beta)$ is obtained as a histogram of $\beta(t_0)$ for all values of $t_0$ and usually it is compared with inverse $\chi^2$, $\chi^2$ (Gamma) and log-normal distributions which have the same mean and variance of $\beta$ (Beck *et al*., 2005).

*3.4. General superstatistical q-parameters*

A general parameter $q$ can be defined as (Beck and Cohen, 2003):

$$q = \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2} \tag{15}$$

where $q$ measures the deviation from Gaussianity and is valid for any superstatistics. When variable $\beta$ does not fluctuate (e.g. constant) then $q = 1$, since $f(\beta) = \delta(\beta\text{-}\beta_0)$. In addition, the $q$ coincides with $q_{stat}$ (*stat* stands for stationary) from Tsallis statistics if $\beta$ follows a $\chi^2$ distribution, namely $q = q_{stat}$=1 + 2/($n$+1), where $n$ is the number of effective degrees of freedom. The Tsallis $q_{stat}$ can be evaluated for $u(t)$ time series using various methods like the one presented in Iliopoulos and Aifantis (2018). On the other hand, if log-normal superstatistics is the case, then $q = q_{log}$= 1/3, where $F$ is the flatness $F = \frac{\langle u^4 \rangle}{\langle u^2 \rangle^2}$ of the distribution $p(u)$. It must be noted here that $q < 1$ is not applicable for superstatistics (Beck, 2004).

*3.5. Additional constraint*

In a recent study (Van der Straeten and Beck, 2009) an extra condition was proposed that should hold before assuming that a time series at hand can be described by superstatistics. If $\theta_{T,l}$ is defined as the deviation of the fourth momentum from $3\langle u^2 \rangle^2_{T,l}$ for the time series slices of length $T$:

$$\theta_{T,l} = \langle u^4 \rangle_{T,l} - 3\langle u^2 \rangle^2_{T,l} \tag{16}$$

then the Gaussian approximation is expected to hold when $|\varepsilon|$<<1, where:

$$\varepsilon = \frac{1}{3} \frac{\sum_{j=1}^{N} \theta_{T,j}}{\sum_{j=1}^{N} \langle u^2 \rangle^2_{T,j}} \,. \tag{17}$$

## 4. Application of superstatistics in Hellenic seismicity

In this study, for the application of superstatistics in seismicity of the Aegean and surrounding area, we consider the interevent times of 1221 earthquakes of $M \geq 5.2$ that occurred in the study area since 1911. The data set used is a subset of the regional earthquake catalogue compiled by the Geophysics Department, Aristotle University of Thessaloniki (http://geophysics.geo.auth.gr/ss/station_index_en.html) and is considered complete at this magnitude threshold. Hereafter, M52

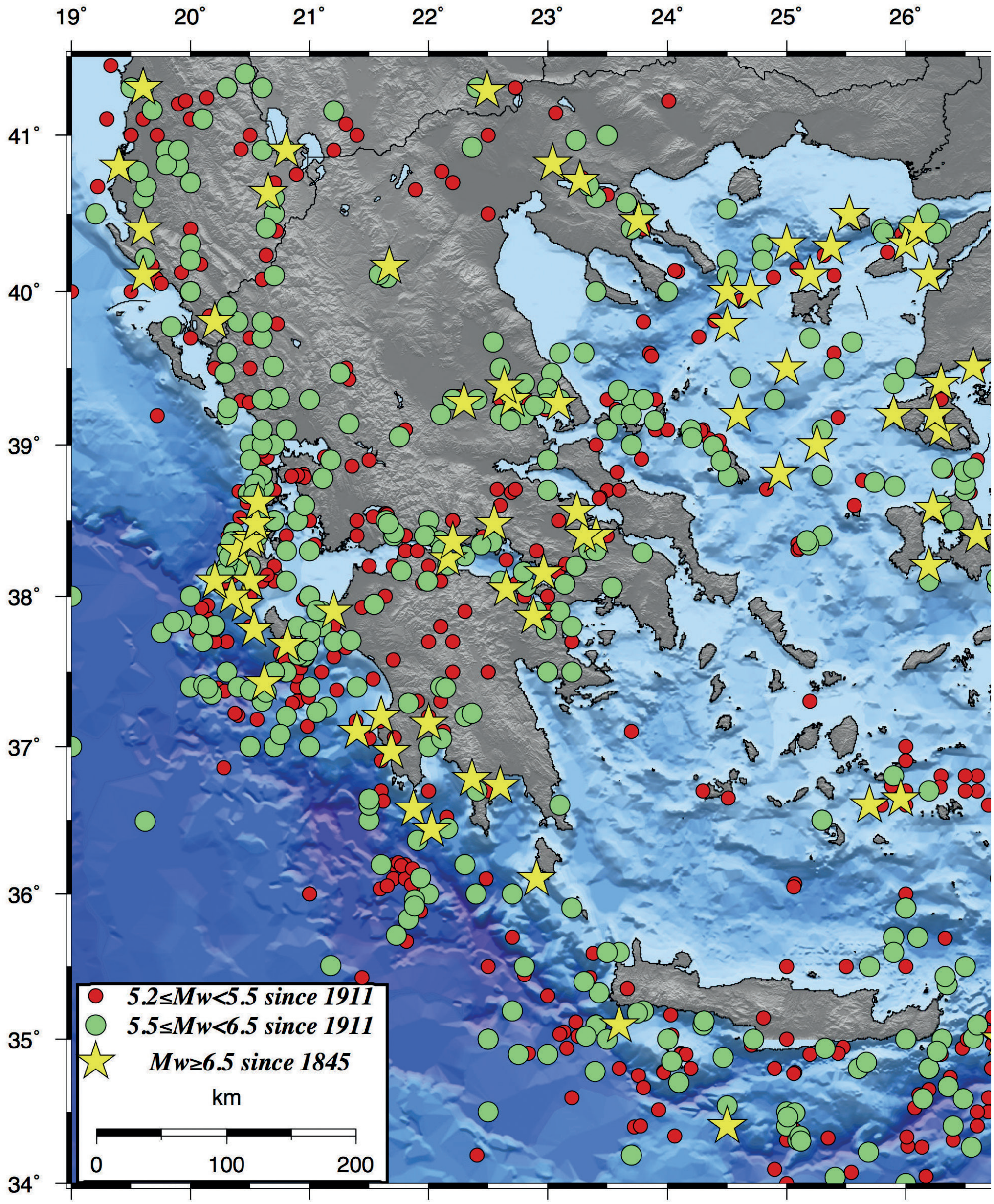


Fig. 3 - Epicentral distribution of earthquakes with magnitudes and occurrence periods considered in this study. In particular, the red circles correspond to earthquakes with $5.2 \leq M < 5.5$, the green to $5.5 \leq M < 6.5$ and the yellow stars to $M \geq 6.5$.

denotes the interevent times of earthquakes with magnitude greater or equal to 5.2.

In Fig. 3 the epicentral distribution of earthquakes considered in this study with magnitudes and occurrence periods given in the legend are shown. In particular, the red circles correspond to epicentral distribution of earthquakes with $5.2 \leq M < 5.5$, the green to $5.5 \leq M < 6.5$ and the yellow stars to $M \geq 6.5$.

Following the descriptions in previous paragraphs, the two time scales $\tau$ and $T$ and their ratio namely $T/\tau$ were estimated, as well as the slow varying intensive parameter $\beta(t)$ was constructed and, then, its probability density with Gamma and log-normal distributions was compared. In particular, using differenced time series $u(t) = \upsilon(t+\delta)-\upsilon(t)$, where $\upsilon(t)$ are the interevent times (or else waiting times) between sequential earthquakes, and $\delta = 2^j$, $j = 0, 1, 2 \ldots, n$ with $n = 7$, eight differenced time series were constructed, one for each $\delta$. In Fig. 4 the original interevent times is presented (Fig. 4a) along with one example of the differenced series (Fig. 4b) corresponding to $\delta = 4$.

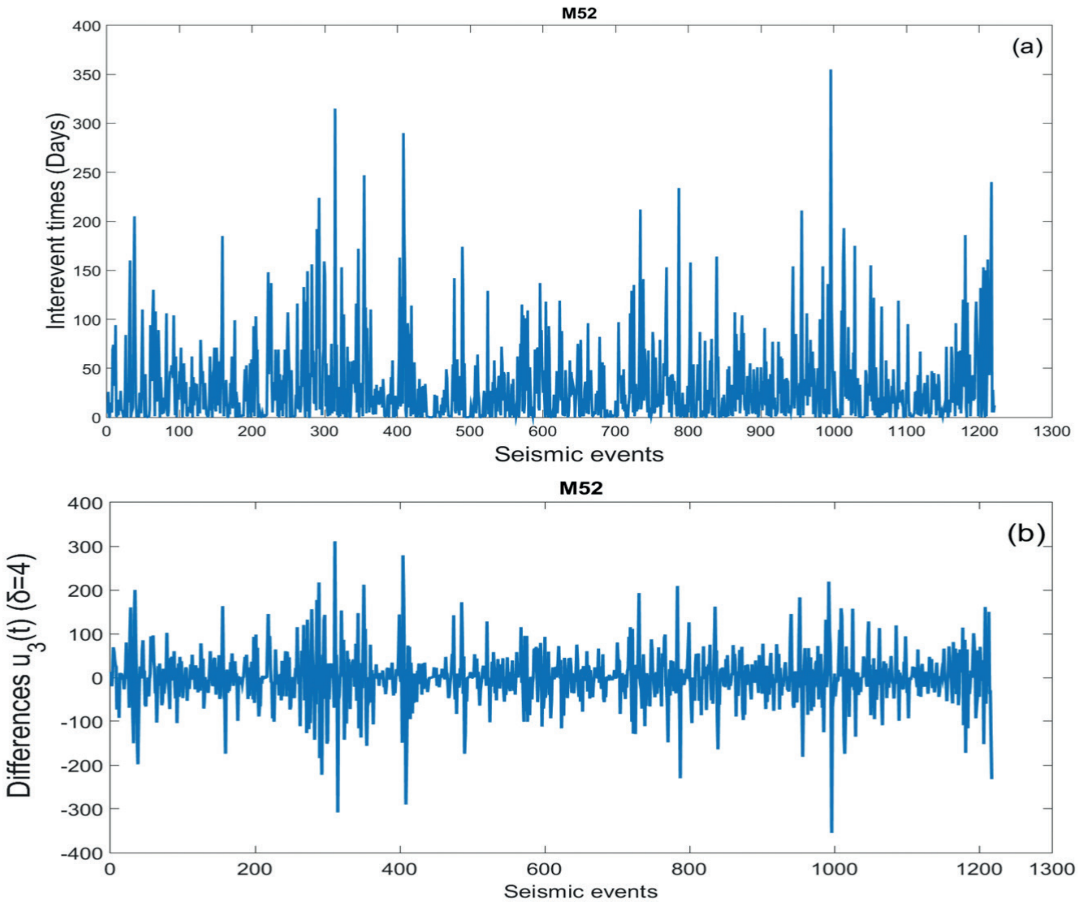


Fig. 4 - a) Interevent time series for earthquakes with $M \geq 5.2$ that occurred in the study area since 1911; b) an example of differenced series $u(t)$ generated with $\delta = 16$ for the interevent time series shown in panel a.

Using Eq. 11 the kurtosis $\varkappa(\Delta t)$ was estimated for each differenced series ($\delta 1$, $\delta 2$, ..., $\delta 128$) and the results are depicted in Fig. 5, where the function $\varkappa(\Delta t) = 1$ when $\Delta t = 1$ and $\varkappa(\Delta t) > 3$ when $\Delta t$ = length of interevent times = 1221, implying intermittency and non-Gaussian behaviour, as expected in superstatistical systems. However, even though in previous papers this function seems to be a smooth linear increasing monotone function (e.g. Beck *et al.*, 2005), in this case $k(\Delta t)$ does not exhibit a profile of this type. In addition, in most differenced series (e.g. $\delta 1$, $\delta 4$, $\delta 16$, $\delta 32$, $\delta 64$) this function oscillates around 3 for a considerable interval of $\Delta t$ = 10-60. This result could be attributed to the short earthquake time series considered and/or to the complexity inherent in the series. All the relevant superstatistical time scales $T$ were estimated based on Eq.

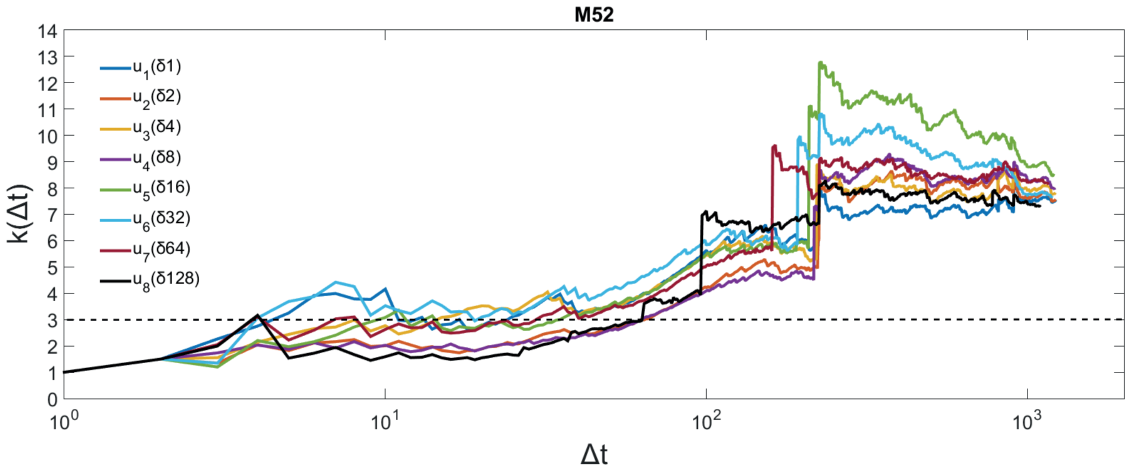


Fig. 5 - Kurtosis $k(\Delta t)$ as a function of $\Delta t$, estimated for all differenced series $u(t)$ generated with $\delta = 1, 2, 4, 8, 16, 32, 64, 128$.

12, namely $\varkappa(T) = 3$, as depicted by the black dotted line in Fig. 5, for each time difference $\delta$. In the cases where function $k(\Delta t)$ fluctuates and equals to 3 multiple times, all relevant time scales $T$ were estimated and the final value considered was the one corresponding to the lowest constraint $\varepsilon$ (see next paragraph). For example, for the differenced series $\delta 4$, the relevant times $T$s were: 8, 14, 15, but, as shown in Table 1, $T = 8$ was kept, since it is associated with the smallest $\varepsilon$.

In order to further validate the results an additional constraint was estimated according to Eq. 17. When this constraint is smaller than one, namely $\varepsilon << 1$ or more loosely $\varepsilon < 1$ for small data sets, then the time windows used for the evaluation of large time $T$ can be assumed to be characterised by Gaussian distributions. Basically, this constraint evaluates the contribution of the deviations from the Gaussian approximation in these time slices to the value of $k(T)$. As it can be seen in Table 1, only two values of constraint ($\varepsilon$) are small enough compared to unity, to validate the superstatistical hypothesis and these values correspond to time differences $\delta = 4, 32$. Small values compared to unity, also were attained for $\delta = 1$ and 64, taking into account the short length of the time series. On the other hand, all the remained values of the constraint reject the hypothesis of superstatistics for the corresponding differences, namely $\delta = 2, 8, 16, 128$ (Van der Straeten and Beck, 2009). The rejection of the superstatistical hypothesis could be due to either the presence in the time series of Gaussian distributed random variables of constant variance with outliers and, if this is not the case, $q_{stat}$-Gaussian (instead of Gaussian) distributed random variables with constant variance (Van der Straeten and Beck, 2009). These cases can falsely be classified as superstatistical (Van der Straeten and Beck, 2009). In addition, as mentioned above, the small length of the original interevent times could also induce spurious results.

In addition, in Fig. 6 the auto-correlation functions for all the differenced time series are presented. As it can be seen, the autocorrelation profiles are not exponential but rather exhibit a fast, abrupt decay towards zero, implying the existence of short auto-correlations. Therefore, for the estimation of short time scale $\tau$, Eq. 13 was used, since there is not any exponential decay, and the results are also given in Table 1. In particular, in Table 1 the values of the superstatistical parameters like the long time scale $T$ corresponding to lowest constraint ($|\varepsilon|$), the constraint ($|\varepsilon|$), the short time scale $\tau$, and their ration $T/\tau$ are presented. As it can be seen the values of $T$ are

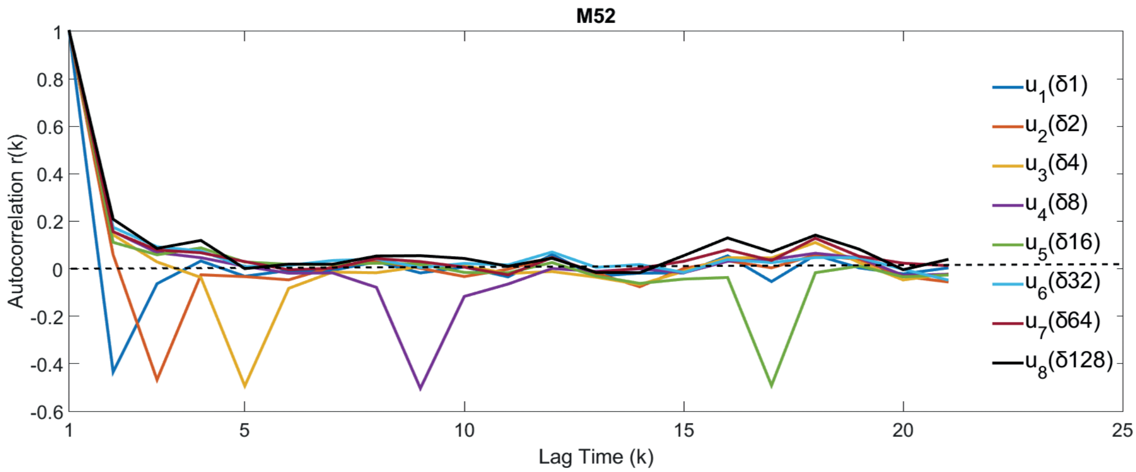


Fig. 6 - Autocorrelation coefficients $r(k)$ estimated as a function of lag time $k$ estimated for the differenced series $u(t)$ ($\delta$ = 1, 2, 4, 8, 16, 32, 64, 128).

larger than the values of $\tau$, and, therefore, the ratio $T/\tau$ attains values much larger than unity, indicating a clear separation of scales, as predicted by superstatistics. However, taking into account the constraint, the superstatistical hypothesis is validated by $\delta$ = 1, 4, 32, 64 differenced series and rejected for the others. Overall, these results reveal a very complex (super)statistical profile for the M52 interevent seismic time series, which strongly depends on the time difference $\delta$ considered for creating the differenced time series. These results also verify the non-smooth fluctuating profile of the $k(\Delta t)$ function as depicted in Fig. 3.

Table 1 - The values of the superstatistical parameters: long time scale $T$, the constraint ($|\varepsilon|$), short time scale $\tau$ and their ratio $T/\tau$ estimated for each differenced series $u(t)$ of the interevent time series $\upsilon(t)$.

| $\delta$ | $T$ | Constraint ($|\varepsilon|$) | $\tau$ | $T/\tau$ |
|---|---|---|---|---|
| 1 | 11 | 0.1856 | 1.41 | 7.80 |
| 2 | 61 | 0.9590 | 1.30 | 46.92 |
| 4 | 8 | 0.0395 | 1.14 | 7.05 |
| 8 | 61 | 1.0778 | 1.20 | 50.83 |
| 16 | 9 | 0.9873 | 1.26 | 7.14 |
| 32 | 4 | 0.0140 | 1.22 | 3.28 |
| 64 | 7 | 0.3185 | 1.23 | 5.70 |
| 128 | 58 | 1.2268 | 1.21 | 47.90 |

Fig. 7 presents an example of slow variable $\beta(t)$ (orange line) and the corresponding differenced time series $u(t)$ (blue). For the generation of $\beta(t)$ we used $\delta$ = 4, $T$ = 8 and Eq. 14. Other generations of $\beta(t)$ are possible for different $\delta$'s and $T$'s. As it can be seen in this figure, the variable $\beta(t)$, changes slowly compared to $u(t)$. This result is also clearly depicted in Fig. 8, that shows the autocorrelation functions estimated for the variable $\beta(t)$ corresponding to differenced series $\delta$ = 1, 4, 32, 64, since only these validated superstatistical hypothesis. As it can be seen, the

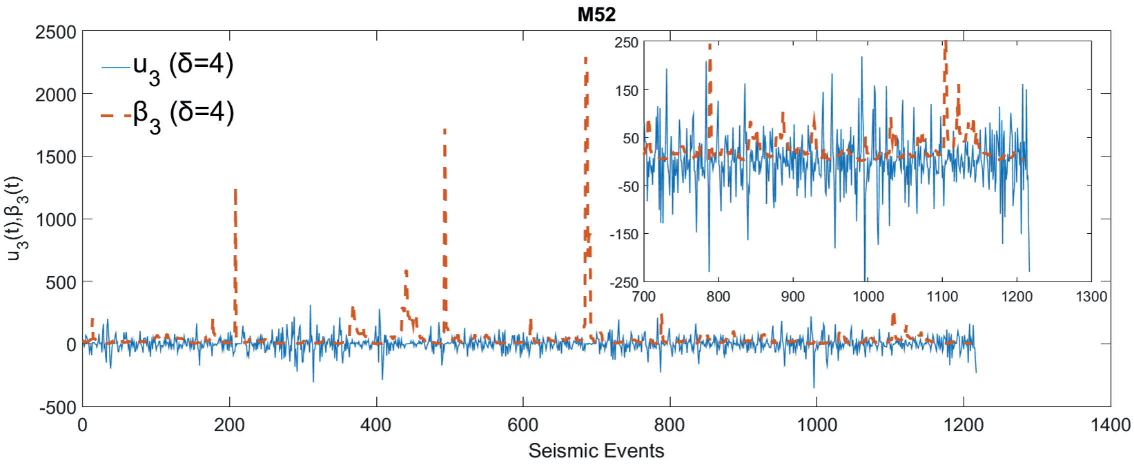


Fig. 7 - Time series of the process $\beta(t)$ (top-orange) and differenced series $u(t)$. For this example $\delta = 4$ and $T = 8$, respectively. As it can be seen $\beta(t)$ is a slow varying process, compared to $u(t)$, as predicted by superstatistical theory.

profile of the autocorrelation functions exhibit a slow decaying profile, indicating a long memory for $\beta(t)$, verifying the superstatistical framework prediction.

Finally, the probability distributions of $\beta(t)$ for the differenced series $\delta$ = 1, 4, 32, 64, are compared with the Gamma and log-normal distributions as described in Section 3.3. Fig. 9 shows the comparison of $f(\beta)$ for beta differenced series corresponding to $\delta = 4$ (red dots), with the log-normal (blue line) and Gamma distribution (green line) in a log-linear plot. The probability density $f(\beta)$ is obtained as a histogram of $\beta(t_0)$ for all values of $t_0$ and is compared with $\chi^2$ (Gamma) and log-normal distributions which have the same mean and variance of $\beta$. As it can be seen in Fig. 9, log-normal distribution fits the original distribution much better especially in the tails. The comparison also took place using two criteria, namely the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) (Aho *et al.*, 2014). For the fitting of Gamma distribution the criteria attained the values: AIC = 8341 and BIC = 8351, whereas for the log-

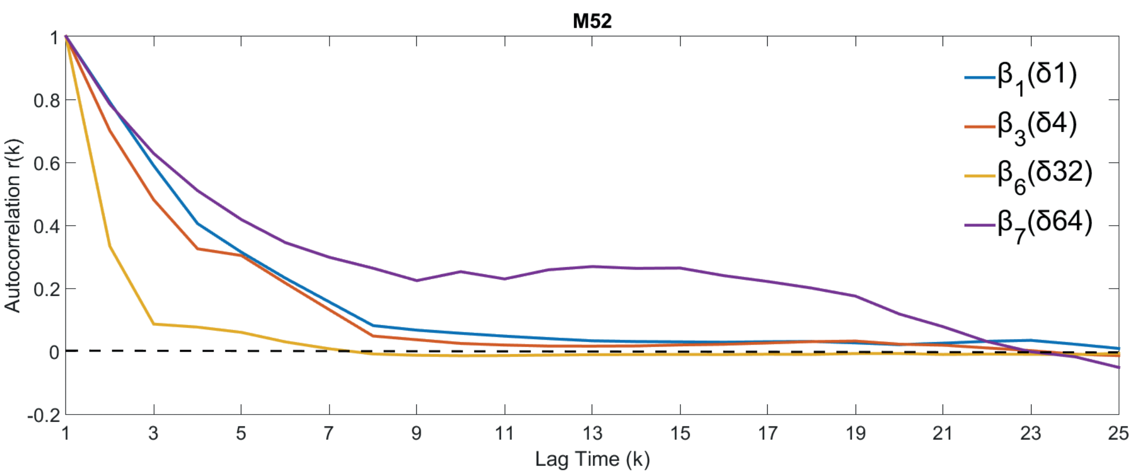


Fig. 8 - Autocorrelation coefficients $r(k)$ estimated as a function of lag time $k$ for the slowly varying variable $\beta(t)$ ($\delta$ = 1, 4, 32, 64).

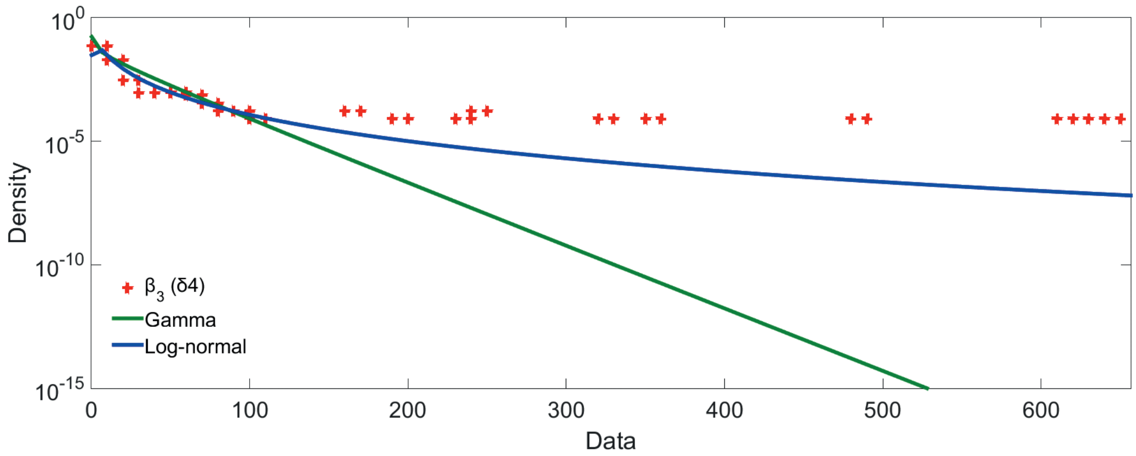


Fig. 9 - Probability density $f(\beta)$ (empirical) extracted from the differenced time series $\delta$ = 4 (red dots) and compared with log-normal (blue line) and Gamma ($\chi^2$) (green line) distributions.

normal distribution: AIC = 7702 and BIC = 7712. The results indicate that the differenced series fall into the log-normal superstatistical class, since both criteria are smaller for the log-normal distribution. Similar results were obtained for the other $\beta(t)$'s too. Therefore, we can claim that these $f(\beta)$ follow the distribution described by Eq. 8 and $u(t)$ the one of Eq. 9, indicating that $\beta$ is a result of the product of $n$ local cascade random Gaussian variables $X$, namely $\beta = \prod_{i=1}^{n} X_i$ and that the respective differences of M52 interevent times are related to a multiplicative turbulent random process (Beck *et al*., 2005).

Summarising, the results of this application reveal:

- fat-tailed non-Gaussian distributions for differenced interevent time series, since $\varkappa(\Delta t) > 3$ for large $\Delta t$. In addition, the profile of $\varkappa(\Delta t)$ deviates from the expected smooth increasing monotone one. This could be due to the short length time series considered inducing fluctuations in the statistical measures and/or to inherent hidden statistical complexity in the data set;
- the presence of two separate time scales, verifying superstatistics theory prediction, since the time scale ratio $T/\tau$ attains values much higher from unity;
- the use of the additional constraint revealed that only the differenced series $\delta$ = 1, 4, 32, 64 validate the superstatistical hypothesis, and not the remained ones. This could be attributed to false superstatistical profile obtained due to the presence of Gaussian distributed random variables of constant variance with outliers or Tsallis $q_{stat}$-Gaussians with constant variance (Van der Straeten and Beck, 2009);
- for differenced M52 interevent times corresponding to $\delta$ = 1, 4, 32, 64, log-normal superstatistics fit better the intensive parameter $\beta$ compared to Gamma distribution. This result implies that $\beta$ is a result of the product of $n$ local cascade random Gaussian variables $X$, namely $\beta = \prod_{i=1}^{n} X_i$. This is in contrast to Tsallis superstatistics where $\beta$ results from an additive process, namely $\beta = \sum_{i=1}^{n} X_i^2$ Therefore, the specific differenced M52 interevent

times are related to a multiplicative turbulent random process, which can be described by local Boltzmann factors $e^{-(\beta u^2)/2}$, whose variance parameter $\beta$ varies slowly according to a log-normal distribution function. The slow variation of parameter $\beta$ was also shown from the estimation of the corresponding autocorrelation functions.

It must be mentioned that the used time series is relatively short and, therefore, additional analysis is required to verify the aforementioned findings, along with the application of Renewal Analysis algorithm (Paradisi *et al.*, 2009), which might be addressed in a forthcoming study.

## 5. Summary and conclusions

The main types of superstatistics are detailed along with the methodology needed for capturing superstatistical features and the corresponding necessary time scales from time series. Real-world applications were also summarised covering a broad range of different subject areas, focusing however in earthquakes. Superstatistics is, thus, a very efficient framework for describing the non-Gaussian behaviour of complex non-equilibrium systems. The superstatistical framework is based on non-equilibrium statistical mechanics and characterises the system under consideration as a superposition of several statistics, which act on different time scales. For example, the system statistics could be derived as a superposition of local Gaussian processes weighted with a process of a slowly fluctuating intensive variance parameter. Overall, superstatistics can provide valuable information both on practical (e.g. time series analysis, modelling), as well as on theoretical studies of complex systems. Themes like extreme value theory (Rabassa and Beck, 2014), fractional kinetics and heterogeneous anomalous diffusion (Itto, 2014), reaction rate theory and fractional calculus (Mathai and Haubold, 2010), and other concepts closely related to complexity theory, can be valuably updated in the prism of superstatistics, leading to advanced theoretical frameworks and mathematical models, for the thorough understanding of complex systems behaviour.

The application of superstatistical methodology investigating the complex turbulent dynamics of seismogenesis in Greece, namely to interevent time series of earthquakes with $M \geq 5.2$, revealed a complex profile for the differenced interevent time series. In particular, depending on the parameter $\delta$ the superstatistical hypothesis can be either verified or not. When the hypothesis is verified, the analysis revealed the log-normal superstatistical character of the differenced interevent time series, indicating a multiplicative turbulent random process. This result deviates from the ones claiming Tsallis superstatical distribution which is related to additive random process of squared Gaussian random variables (Beck *et al.*, 2005). When the superstatistical hypothesis is not verified, the results can be related to Gaussian distributed random variables of constant variance with outliers or Tsallis $q$-Gaussian variables with constant variance, as indicated in Van der Straeten and Beck (2009). However, a more extensive and thorough study concerning superstatistics and Hellenic seismicity, for various magnitude thresholds and time intervals, needs to verify the aforementioned results and this will be the topic of a future paper.

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