

## Combining EGM2008 with GOCE gravity models

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**ABSTRACT** The most advanced global gravity model, EGM2008, is nowadays competing with local models in terms of resolution and accuracy in the definition of the Earth gravity field. This global model, complete to spherical harmonic degree and order 2159, is however affected by several biases mainly due to datum inconsistencies and variability of the input observations density and accuracy. This article addresses the problem of improving the EGM2008 model exploiting a satellite-only model based on the Gravity and steady state Ocean Circulation Explorer (GOCE) mission. The GOCE model, apart from being more accurate in the medium frequencies, is not affected by local biases since it is obtained by a global homogeneous data set referred to a unique geocentric ellipsoid: so both effects of different data sources and inconsistent height datums are not present. The resulting combination can vary depending on the weighting of the two global models. A first simple solution is to average spherical harmonic coefficients of the same degree and order by computing the weights as the inverse of their error variance. Another attempt is to consider for the GOCE model also the error correlations of the coefficients that are available through an order-wise covariance matrix. The paper shows also a way to integrate the two available, but not fully consistent, sources of information about the EGM2008 error, i.e., the spherical harmonic coefficient variances and a geographical estimate of the geoid variance. The study has been performed on the Mediterranean area because it was required for the GOCE-Italy project and because in this area the obtained merged geoid can be validated, e.g., by using available drifter data. The main conclusion is that there is no a general criterion to chose which is the optimal way to merge the EGM2008 and GOCE models. This is because the full stochastic structure of EGM2008 is not available and only approximations can be used. This means that each case study has to be dealt with separately. In particular for the Mediterranean area the differences between the proposed combinations are here presented.

**Key words:** Earth gravity field, GOCE, EGM2008, geoid combination.

### 1. Introduction

The problem of combining different types of information to derive the “best” model of the Earth gravity field is a crucial item, keeping geodesists busy in the last decades. This is indeed shared by all other geophysical disciplines.

Up to now different methods have been developed to produce global gravity models [see for example Lerch *et al.* (1972), Rapp (1975, 1984), Balmino *et al.* (1976), Wenzel (1985), Lemoine *et al.* (1998), Tscherning (2001), Reigber *et al.* (2005), Bruinsma *et al.* (2010), Pavlis *et al.* (2012)] and, somewhat separately, local gravity models. In the latter case, the hope has always been that numerous, detailed and accurate local observations on the gravity field could be used to improve the global representation which is by force restricted in terms of resolution, because of inhomogeneities of the data and computing limitations. Several solutions to this problem are known in geodetic literature such as the “localization” of Molodensky’s theory (Molodensky *et al.*, 1962), the Taylorizing of global models to local data (e.g., Wenzel, 1982; Kearsley and Forsberg, 1990; Reguzzoni *et al.*, 2011), the Stokes-Helmert approach (Stokes, 1849) and its variants [between the others: Vaníček and Sjöberg (1991), Martinec and Vaníček (1994), Heck (2003), Novak (2007)], the collocation approach (Moritz, 1980; Sansò, 1986; Tscherning, 1994; Krarup, 2006) with its particular form of remove-restore principle (Forsberg, 1994) or more recently a method based on radial basis functions (Klees *et al.*, 2008).

However at the present level the most advanced global gravity field model, Earth Gravity Model 2008 (EGM2008) (Pavlis *et al.*, 2012), is in part competing with local models in terms of resolution and accuracy (Pacino and Tocho, 2009; Hirt *et al.*, 2011). This means that the finite space in which EGM2008 is computed, basically solid spherical harmonics complete to degree and order 2159, is large enough to leave out an omission error that is really so small to be negligible in most areas of the world (Pavlis *et al.*, 2012), especially those where a good data coverage is provided. This is not the case in other areas like Africa, South America or Antarctica (Bomfim *et al.*, 2013). Let us specify here that in this work the focus is on the combination of global models for the prediction of the geoid. When other functional of the gravitational potential (e.g., higher derivatives) are considered the omission error of high and very high frequencies can become significant as explained for instance in Hirt (2010).

So two questions seem to be critical nowadays:

- a) to improve EGM2008 because of several biases, still present in the low-medium frequency band (below degree 240), due to datum inconsistency, variability of the measurement density and accuracy and differences in the reduction of the available data set;
- b) to improve local solutions by means of new local data.

The present paper is working on item a) exploring a few solutions that have been developed within the GOCE-Italy project. The item b) although object of proposals (Pail *et al.*, 2010), is certainly difficult since, in the opinion of the authors, it requires developing a new theory and algorithms allowing a non-homogeneous, non-isotropic representation of the local gravity field.

Also on the item a) proposals and experiments are present in literature [see: Schuh (1996), Koch and Kusche (2002), Reguzzoni and Sansò (2012), Gatti *et al.* (2013)]. Yet the lack of a rigorous error covariance information requires a certain degree of empirical and numerical experimentation, the results of which represent the main objective of the article. In this respect the authors are aware that the problem of EGM2008 biases is not directly addressed in this work, however the low harmonic degrees ( $\leq 70$ ) are not significantly affected by these systematic errors being estimated by CHAMP and GRACE satellite missions mainly. At medium wavelengths (where the biases effect is bigger) the combination is dominated by the unbiased GOCE model (see error variances in Fig. 2) so also the biases effect, even if not directly faced,

is reduced. Finally at higher frequencies the residual biases effect is probably negligible [in the order of 1 cm as has been verified for the height datum problem in Gatti *et al.* (2013)].

## 2. Available stochastic information of EGM2008 and GOCE-based GGMs

The model EGM2008 extends to spherical harmonic degree 2190 (selected orders only) as a result of transforming an ellipsoidal model of degree/order 2159 to a spherical one (Jekeli, 1988). The model is computed as a solution of the Molodensky problem from a global set of area-mean free-air gravity anomalies. In particular, it implies to transform geoid undulation from altimetry in free air gravity anomalies offshore and to downward continue gravity anomalies from the Earth surface to the ellipsoid in continental area. This last step is performed via a smoothing through a residual terrain correction and a least squares collocation. Once the ellipsoid has been covered by a regular grid of mean gravity anomalies  $\Delta g$  values, the spherical harmonic coefficients are obtained via least squares up to some harmonic degree and then by integration. Lower degrees (below degree 70) are then integrated with the direct information coming from the satellite-to-satellite tracking of the GRACE mission (Mayer-Gürr, 2006). More details with a thorough discussion of data and methods can be found in Pavlis *et al.* (2012).

The point however here is that in order to get a feasible least squares solution one has to assume that the noise variance in  $\Delta g$  is only latitude dependent, i.e., constant along parallel stripes on the ellipsoid. This in fact brings the shape of the normal matrix to a block diagonal form, which is then numerically manageable (Colombo, 1981; Sansò and Tscherning, 2003; Reguzzoni *et al.*, 2011). In Reguzzoni and Sansò (2012), it is clearly shown, by means of a numerical example, that this non-optimal block diagonal solution, as well as a numerical integration solution, does provide estimates very close to the optimal ones. This is also confirmed by the excellent performances of EGM2008 reported for instance by the EGM2008 evaluation team in Newton's Bulletin (EGM2008 evaluation team, 2009).

Nevertheless this leaves us with a poor information on the error propagation to the estimated spherical harmonic coefficients. In fact the inverse of the blocks are not representative of the true estimation error covariance, because the actual covariances of the observations have not been used in building the normal matrix. As a matter of fact, stripes passing through the Himalayas or the Andes etc., have a strong variation in error variance of  $\Delta g$  due to both the intrinsic geographic variability and the kind of preprocessing used to reduce the data on the ellipsoid. Moreover the quadrature formulas allow to compute an approximate error propagation (Pavlis and Saleh, 2005) but a full covariance propagation becomes numerically unfeasible.

Monte Carlo approach will be probably able to improve our knowledge of the actual covariance of the EGM2008 model (Gundlich *et al.*, 2003; Koch, 2005; Alkhatib and Schuh, 2007), however the current situation is that the available information on the error structure of the coefficients of EGM2008 is only in terms of error variances  $\sigma_{nm}^2$  of the individual coefficients of degree  $n$  and order  $m$ . This is certainly a valuable information, especially when global combinations at the level of coefficients have to be performed, but it is not a reliable representation of the geographical distribution of the error. In fact, note that a diagonal structure of the variance-covariance matrix of the error, when the further symmetry:

$$\sigma^2(T_{nm}) = \sigma_{nm}^2 \quad \sigma_{n,-m}^2 = \sigma_{n,m}^2 \tag{1}$$

is assumed for the coefficients  $T_{nm}$ , produces a geographical error in the geoid undulation  $N$  that is constant along parallels. In fact being the relation between  $N$  and the disturbing potential  $T$  given by (Heiskanen and Moritz, 1967):

$$N = \frac{T}{\gamma} \tag{2}$$

where  $\gamma$  is the normal gravity, it follows that the error covariance between two points  $N_1 = N(\vartheta_1, \lambda_1)$  and  $N_2 = N(\vartheta_2, \lambda_2)$  is (Reguzzoni and Sansò, 2012):

$$\begin{aligned} \sigma(N_1, N_2) = & \frac{1}{\gamma^2} \sum_{n,m} \bar{P}_{n,|m|}(\vartheta_1) \bar{P}_{n,|m|}(\vartheta_2) \\ & \cdot (\sigma_{nm}^2 \cos m\lambda_1 \cos m\lambda_2 + \sigma_{n-m}^2 \sin m\lambda_1 \sin m\lambda_2) \\ & \frac{1}{\gamma^2} \sum_{n,m} \bar{P}_{n,|m|}(\vartheta_1) \bar{P}_{n,|m|}(\vartheta_2) \sigma_{nm}^2 \cos m(\lambda_1 - \lambda_2). \end{aligned} \tag{3}$$

Obviously in the case of variance, it comes out:

$$\sigma^2(N) = \frac{1}{\gamma^2} \sum_{n,m} \sigma_{nm}^2 \bar{P}_{n,|m|}^2(\vartheta). \tag{4}$$

Actually, even if Eq. (1) is not strictly respected in EGM2008, the resulting variability of  $\sigma^2(N)$  along a parallel is not realistic. This is shown in Fig. 1 where  $\sigma^2(N)$  as a function of longitude  $\lambda$  is displayed along the equator: as one can easily see not even the change land-ocean has a correlation with the plotted variance.

Conscious of this drawback, the authors of EGM2008 have computed by a patient ad hoc work, using independent data, a geographically meaningful estimate of  $\sigma^2(N)$ , that from now on we will denote as  $\sigma_N^2(\vartheta, \lambda)$ . This is available at the EGM2008 website (<http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008>). We underline that this is a precious source of error information, which is not contained into  $\sigma^2(T_{nm})$ .

Coming now to the GOCE gravity model, we first of all underline that the ESA-GOCE mission has not yet concluded its operational phase, so that the final result is not yet available. However partial results are already excellent and we can say that the GOCE model will improve on EGM2008, at least in the medium-low frequencies: improvements up to degree and order 240 are expected in region with poor ground data coverage, e.g., Himalaya, South America and Africa [Pail et al. (2011) or Rummel et al. (2011)], while in area with high dense data coverage the expected improvement will be focused on lower degrees [Hirt et al. (2011) or Šprlák et al. (2011)]. As it is well known there are different solutions of the gravity model from the GOCE gradiometer (Pail et al., 2011). In particular, the space-wise solution (Migliaccio et al., 2004;

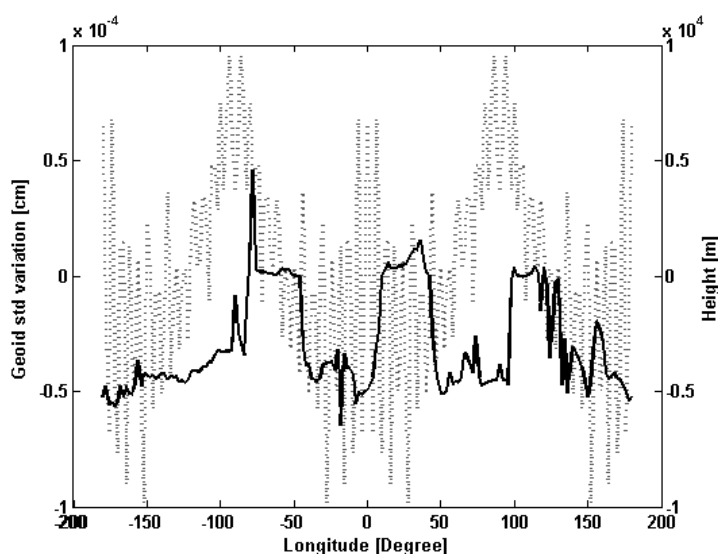


Fig. 1 - Variability of geoid error standard deviation along the equator considering different accuracy for the sine and the cosine coefficients (dashed grey line) and orthometric heights (solid black line).

Reguzzoni and Tselfes, 2009) is available to the authors with its full covariance matrix. To be precise the second release of the space-wise solution is considered in this study (Migliaccio *et al.*, 2011). This is only an approximate, but we deem quite reliable, error information achieved by a Monte Carlo technique (Migliaccio *et al.*, 2009).

The GOCE error covariance matrix  $C_{TT}$ , with  $\underline{T}$  the vector of the anomalous potential coefficients sorted by increasing order  $m$ , all degrees  $n \geq m$ , has been studied and proved to be well approximated by a block-diagonal form [see: Pertusini *et al.* (2010)] which is much more manageable for our purposes.

So, concluding the section, we could say that our information coming from the GOCE mission is a set of  $T_{nm}$  coefficients complete to order and degree 240 and of the relative error covariance matrix approximated in a block-diagonal form.

Would EGM2008 have an analogous covariance information to GOCE, one could apply the model combination discussed for example in Reguzzoni and Sansò (2012), but since this is not the case we have to resort to empirical solutions.

### 3. The direct computation of combined coefficients

To summarize the discussion of Section 2, we could say that the information available is:

$$T_{nm}^E = T_{nm} + \varepsilon_{nm}^E, \quad |m| \leq n, 2 \leq n \leq 2159 \tag{5}$$

$$\sigma^2(\varepsilon_{nm}^E) = \Sigma_{nm}^E \tag{6}$$

$$T_{nm}^G = T_{nm} + \varepsilon_{nm}^G, \quad |m| \leq n, 2 \leq n \leq 240 \tag{7}$$

$$\sigma(\varepsilon_{nm}^G, \varepsilon_{jk}^G) = C_{nm,jk}^G \tag{8}$$

where  $T_{nm}^E$  are the coefficients of  $T$  in EGM2008,  $\varepsilon_{nm}^E$  their stochastic errors, if we assume that  $T_{nm}^E$  are bias free;  $T_{nm}^G$  are the coefficients derived from GOCE data and  $\varepsilon_{nm}^G$  their errors.

Note that, if we vectorize  $\{T_{nm}\}$  into a vector  $\underline{T}$  and correspondingly the  $\{\varepsilon_{nm}^E\}$  and  $\{\varepsilon_{nm}^G\}$ , we can claim that whatever is the order chosen:

$$C(\underline{\varepsilon}^E, \underline{\varepsilon}^E) = K = \Sigma^E + R, \tag{9}$$

where  $\Sigma^E$  is the known diagonal of the covariance matrix, while  $R$  is the unknown off-diagonal part:

$$C(\underline{\varepsilon}^G, \underline{\varepsilon}^G) = C^G. \tag{10}$$

If for the first 240 degrees we choose a progression of  $T_{nm}$  by orders, i.e.,  $m = 0, m = \pm 1, \dots, \pm 240$ ,  $C^G$  is block-diagonal, namely:

$$C_{nm,jk}^G = \delta_{mk} B_{nj,m}^G, \quad m \leq (n, j) \leq 240, \tag{11}$$

where  $B_{nj,m}^G$  is the full covariance matrix and  $\delta$  is the Dirac function. This is the information, so to say, at the level of coefficients. On the other hand we have also available, for a given grid of geoid values,  $N(\vartheta, \lambda)$ , the variances  $\sigma^2(\vartheta, \lambda) = \sigma^2 [N(\vartheta, \lambda)]$ .

The first idea to combine the two models then could be to use the lowest level of information we have, namely the individual variances  $\Sigma_{nm}^E = \sigma^2(\varepsilon_{nm}^E)$  and  $B_{nm,m}^G = \sigma^2(\varepsilon_{nm}^G)$ ; with this information only, the minimum variance linear estimation is just the weighted mean of the two data, namely:

$$\tilde{T}_{nm} = \left[ \frac{1}{\Sigma_{nm}^E} + \frac{1}{B_{nm,m}^G} \right]^{-1} \left[ \frac{T_{nm}^E}{\Sigma_{nm}^E} + \frac{T_{nm}^G}{B_{nm,m}^G} \right], \quad 2 \leq n \leq 240, \quad |m| \leq n \tag{12}$$

Such an estimate has the well-known variance:

$$\sigma^2(\tilde{T}_{nm}) = \left[ \frac{1}{\Sigma_{nm}^E} + \frac{1}{B_{nm,m}^G} \right]^{-1}. \tag{13}$$

We notice that since each  $T_{nm}$  comes only from the two corresponding coefficients in EGM2008 and GOCE, the combined model will completely agree with EGM2008 above degree 240, without any modification of higher degree coefficients, as it happens on the contrary if EGM2008 would have a known block-diagonal covariance [see: Reguzzoni and Sansò (2012)].

The result of Eq. (13) globally gives rise to Fig. 2 in terms of geoid error degree variances. It can be clearly seen that within this solution EGM2008 model is influenced by GOCE solution only in the range of harmonic degrees between 40 and 180. Outside this range the model is completely unchanged.

The second attempt one can think of, could be to use at least that part of the stochastic

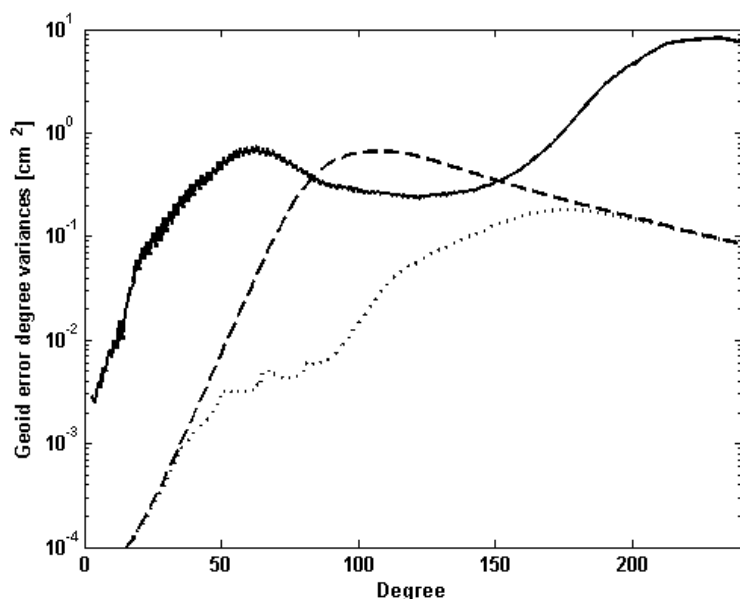


Fig. 2 - Geoid error degree variances [cm<sup>2</sup>] for EGM2008 (dashed line), GOCE (solid line) and the merged model (dotted line).

information that is contained into Eq. (11), i.e., all the matrix  $C^G$ . This means considering the error of the vector  $\underline{T}^G$  correctly correlated, but the error of  $\underline{T}^E$  only diagonal, in lack of a better information. In this case the newly estimated coefficients by the least squares principle are given by a combination, weighted by the covariance matrices order by order up to  $n^G = 240$ , the maximum degree present in the GOCE solution. So if we put:

$$\underline{T}_m^G = \{T_{nm}^G; m \leq n \leq n^G\} \tag{14}$$

$$B_m^G = \text{covariance of the error of } \underline{T}_m^G \tag{15}$$

$$\tilde{\underline{T}}_m^E \equiv \{T_{nm}^E; m \leq n \leq n^G\} \tag{16}$$

$$\tilde{\tilde{\Sigma}}_m^E \equiv \text{diagonal part of the covariance of the error of } \tilde{\underline{T}}_m^E \tag{17}$$

we get the estimators:

$$\tilde{\underline{T}}_m \approx \left[ \left( \tilde{\tilde{\Sigma}}_m^E \right)^{-1} + \left( B_m^G \right)^{-1} \right]^{-1} \left[ \left( \tilde{\tilde{\Sigma}}_m^E \right)^{-1} \tilde{\underline{T}}_m^E + \left( B_m^G \right)^{-1} \underline{T}_m^G \right], \quad m = 0, 1, \dots, n^G = 240. \tag{18}$$

The covariance matrices of  $\tilde{\underline{T}}_m$  would be given by:

$$\tilde{\tilde{C}}_m = \left[ \left( \tilde{\tilde{\Sigma}}_m^E \right)^{-1} + \left( B_m^G \right)^{-1} \right]^{-1}, \tag{19}$$

if the stochastic model leading to Eq. (18) would have been correct; but this is not the case.



So, the true covariance of  $\tilde{T}_m$  can only be obtained by covariance propagation. If we denote as  $\tilde{\Sigma}_m^E + R_m$  [recall Eq. (9)] the covariance of the error of  $\tilde{T}_m^E$ , then one gets:

$$C\left(\tilde{T}_m, \tilde{T}_m\right) = \left[ \left(\tilde{\Sigma}_m^E\right)^{-1} + \left(B_m^G\right)^{-1} \right]^{-1} \left\{ \left(\tilde{\Sigma}_m^E\right)^{-1} \left(\tilde{\Sigma}_m^E + R_m\right) \left(\tilde{\Sigma}_m^E\right)^{-1} + \left(B_m^G\right)^{-1} B_m^G \left(B_m^G\right)^{-1} \right\} \left[ \left(\tilde{\Sigma}_m^E\right)^{-1} + \left(B_m^G\right)^{-1} \right]^{-1} \tag{20}$$

After some elementary manipulations, Eq. (20) can be cast into a more interesting form, namely:

$$C\left(\tilde{T}_m, \tilde{T}_m\right) = \left[ \left(\tilde{\Sigma}_m^E\right)^{-1} + \left(B_m^G\right)^{-1} \right]^{-1} + B_m^G \left(\tilde{\Sigma}_m^E + B_m^G\right)^{-1} R_m \left(\tilde{\Sigma}_m^E + B_m^G\right)^{-1} B_m^G. \tag{21}$$

Since the matrices  $R_m$  are not definite (neither positive nor negative), as they have all 0 entries in the main diagonal, implying that the sum of their eigenvalues has to be zero too, one cannot a-priori conclude that  $\tilde{T}_m$  is better than the simple  $\tilde{T}_m$  derived from Eq. (12).

Remark: in order to come to a clearer statement on the above question, we have been able to construct a counterexample where it is shown that if we have two positive definite covariance matrices  $K, C$  and we combine two “observations” of the same vector  $x$  by using only  $C$  and the diagonal part of  $K$ , as in Eq. (17), there are cases when the components of the combined estimate have a variance larger than those obtained by a simple weighted average, like in Eq. (13). The proof can be found in the Appendix.

This says that one has to be very careful (and maybe conservative) when combining data of an uncertain stochastic structure.

#### 4. On the “local” combination of global models

In an attempt to improve the solution proposed in the previous section, the authors have sought a procedure capable of including the “local” information provided by the point-wise geoid error  $\sigma_N(\vartheta, \lambda)$  (see: Section 2).

To arrive at a feasible computation we were forced to work at a local level, what we did for the test area of the Mediterranean Sea. The idea is as follows: let us choose a particular functional of  $T(P)$ , that we want to know at best in a given local area. In our case, to fix the ideas, we choose the geoid undulation  $N(\vartheta, \lambda)$  as defined in Eq. (2). With the GOCE model global coefficients,  $T^G$ , we can compute  $N^G$  at the nodes of a regular grid, namely, with obvious notation:

$$N\left(\vartheta_i, \lambda_j\right) = \frac{1}{\gamma_{ij}} \sum_{n=2}^{240} \sum_{m=-n}^n T_{nm}^G Y_{nm}\left(\vartheta_i, \lambda_j\right). \tag{22}$$



where  $Y_{nm}$  are the spherical harmonic functions. Eq. (22) can be put in vector form as:

$$\underline{N}^G = A^G \underline{T}^G, \tag{23}$$

and, knowing the error covariance of  $\underline{T}^G$ , the full error covariance of  $\underline{N}^G$  can be computed by covariance propagation, i.e.:

$$C_{NN}^G = C(\underline{N}^G, \underline{N}^G) = A^G C(\underline{T}^G, \underline{T}^G) (A^G)^T. \tag{24}$$

The idea now would be to do the same with the EGM2008 coefficients solution, however we do not have a correct information on  $C(\underline{T}^E, \underline{T}^E) = C^E$ . Furthermore, the error descriptions we have are contradictory to one another. In fact if we take  $\Sigma^E$  (only diagonal) as covariances of the errors of  $\underline{T}^E$ , we can compute the variances of  $N^E(\vartheta_i, \lambda_j)$  by propagation (Fig. 3) and realize that they are quite different from the local covariances  $\sigma_N^2(\vartheta_i, \lambda_j)$  taken from the EGM2008 website (<http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008>) and shown in Fig. 4.

One idea then could be to try to keep the correlation structure of  $\underline{N}^E$ , as derived from the variances of the individual coefficients  $\underline{T}_{nm}^E$ , but to impose to the variances of  $N^E(\vartheta_i, \lambda_j)$  to be equal to the local variances  $\sigma_N^2(\vartheta_i, \lambda_j)$ . In principle this computation is done by the formula:

$$C^E(N_{ij}, N_{lk}) = \frac{1}{\gamma_{ij}\gamma_{lk}} \sum_{n,m} \Sigma_{nm}^E Y_{nm}(\vartheta_i, \lambda_j) Y_{nm}(\vartheta_l, \lambda_k) \tag{25}$$

$|m| \leq n \quad n \leq 2159,$

that provides the covariance matrix:

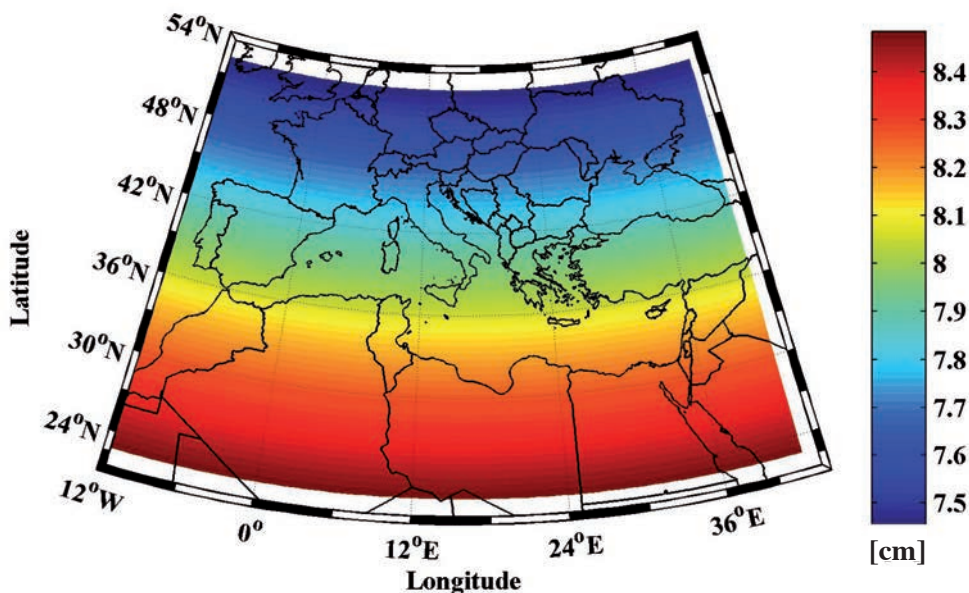


Fig. 3 - Geoid error standard deviation in the Mediterranean area computed from EGM2008 global model coefficients variances, till degree 2159.

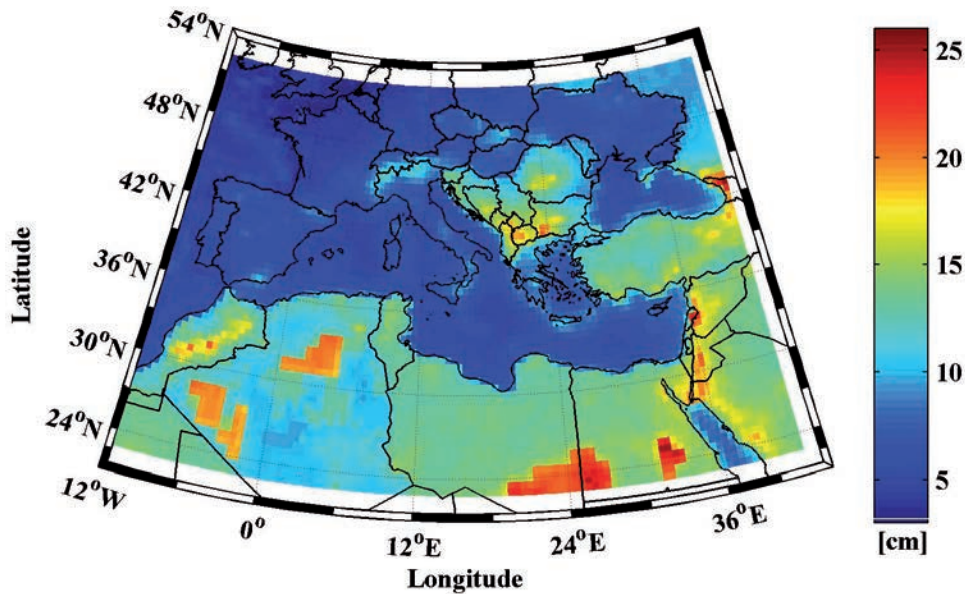


Fig. 4 - Geoid error standard deviation in the Mediterranean area taken from the map delivered from the author of the EGM2008 model. It refers to full resolution, i.e., degree 2159.

$$C_{NN}^E = C(\underline{N}^E, \underline{N}^E). \tag{26}$$

Now let  $D^E$  be a diagonal matrix such that:

$$(D^E)^2 = \text{diagonal } C_{NN}^E; \tag{27}$$

then of course:

$$R^E = (D^E)^{-1} C_{NN}^E (D^E)^{-1} \tag{28}$$

is the correlation matrix relative to the covariance  $C_{NN}^E$ . In order to come to a “localized” covariance of  $\underline{N}^E$  one can take a diagonal matrix  $S^E$  such that the elements of its diagonal are just the standard deviations  $\sigma_N(\vartheta_p, \lambda_j)$ . Therefore the new matrix:

$$L_{NN}^E = S^E (D^E)^{-1} C_{NN}^E (D^E)^{-1} S^E \tag{29}$$

is the sought “localized” covariance of  $\underline{T}^E$

Finally we have to observe that the computation of Eq. (26) for the full EGM2008 model is not a simple task; however we have to correct  $\underline{T}^E$  by means of  $\underline{T}^G$  only below degree  $n_G=240$ , where  $\underline{T}^G$  is really providing new information. Therefore we could say that from  $\underline{T}^E$  we can extract only the  $\{2 \leq n \leq 240, |m| \leq n\}$  part, namely the vector that in Section 3 we denoted  $\tilde{\underline{T}}^E$  and consequently compute  $\underline{N}^E$  and its covariance  $\tilde{C}_{NN}^E$ . The diagonal part  $\tilde{D}^E$  then follows at once. A little more ambiguous is how to choose the diagonal elements of  $S^E$ , taking into account that they have to express the local standard deviations  $\tilde{\sigma}_N(\vartheta_p, \lambda_j)$ , reduced to the error of the first 240 degree only. In this respect the only reasonable solution we found has been to scale the local

errors  $\sigma_N(\vartheta_i, \lambda_j)$  by a factor  $\rho$ , which is just the ratio of the commission error of EGM2008 up to degree 240 with respect to the commission error of the complete model. This is computed from the  $\sum_{nm}^E$  variances, associated to each coefficient. In other words we have put:

$$\tilde{\sigma}_N = \rho \sigma_N \quad (30)$$

$$\rho = \frac{\sum_{n=2}^{240} \sigma_{nm}^2}{\sum_{n=2}^{2159} \sigma_{nm}^2} \quad (31)$$

in the Mediterranean area. Once the error covariance matrices have been properly settled, the combined geoid can be computed by means of standard collocation (Pail *et al.*, 2010).

Note that the Mediterranean area has been selected because this was one of the aims of the GOCE-Italy project, but also because in this area there is a data set of mean sea surface velocities derived from a completely independent source of data, namely the drifter data. As we know the mean dynamic topography ( $\eta$ ), equal to the difference between the mean sea surface (MSS) and the geoid ( $N$ ), and the water velocities under a geostrophic hypothesis ( $V_G$ ), are linked by the relation:

$$k \times f V_G = -g \nabla \eta \quad (32)$$

where  $g$  is the acceleration due to gravity,  $f$  the Coriolis parameter,  $k$  the vertical unit vector (Maximenko *et al.*, 2009). Given a MSS model, a discretized form of Eq. (32) can be used to test an  $N$  geoid model. The interested reader can find the comparison between the different combinations presented in this work and drifter data in the Mediterranean area in another paper of this volume.

## 5. Results and conclusions

In this section we will present some numerical results obtained by applying the combination procedures described before. The area of interest extends around the Mediterranean Sea, including also surrounding countries and in particular northern Africa where significant corrections to the EGM2008 model are expected due to lack of data. The Alpine area is also considered since it is of interest for the GOCE-Italy project. To be more precise the study area is defined by these limits: from 23° N to 52° N in latitude and from 14° W to 42° E in longitude.

Fig. 5 shows the geoid obtained by synthesizing the EGM2008 global gravity model up to the maximum degree available for the space-wise GOCE model, which is 240. It is clear that the geoid undulation is smoother than the one obtainable by exploiting the full EGM2008 model that goes till degree 2159. We have decided to show just the low frequency component of the model (till degree 240) because this is where GOCE contributes to improve EGM2008.

The difference between the geoid computed with GOCE and EGM2008 in the Mediterranean

area is illustrated in Fig. 6. They are of the order of 0.29 m in terms of standard deviation, with the highest differences concentrated in the Middle East area, where probably EGM2008 information is weaker. To evaluate the impact of the GOCE model on EGM2008, the statistics related to the different combination procedures are reported in Table 1. The combination based on error coefficient variances is called EGM08\_GOCE\_CV, the one based on GOCE error block covariances is called EGM08\_GOCE\_BC, while the solution locally adapted to the EGM2008 point-wise geoid error variances is named EGM08\_GOCE\_BC\_MED.

Table 1 - Statistics of the differences between the GOCE based solutions and the EGM2008 geoid up to degree 240, in metres.

Model	Min (m)	Max (m)	Mean (m)	Std (m)
GOCE-only	-1.733	1.661	-0.0007	0.287
EGM08_GOCE_CV	-0.913	0.856	0.0002	0.107
EGM08_GOCE_BC	-0.930	0.888	0.0004	0.114
EGM08_GOCE_BC_MED	-1.541	0.807	-0.0002	0.096

The difference between the use of the simple coefficient variances or the more complete order-wise block covariances for the GOCE error is small (of the order of 0.074 m in standard deviation), as can be also seen in Fig. 7. Comparing the solutions both using block covariances from GOCE, but introducing or not the local adaptation to the EGM2008 geoid error variances, see Fig. 8, one can see that the differences are larger where EGM2008 error is expected to be higher (see also Fig. 4). This is exactly the purpose of this further step of local adaptation.

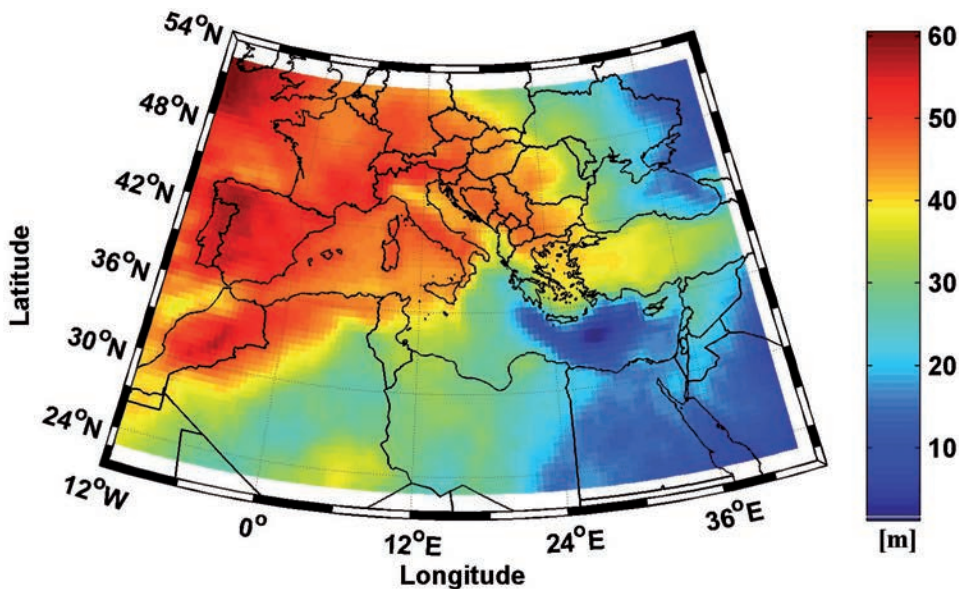


Fig. 5 - Geoid model obtained with a synthesis of the EGM2008 global model till degree 240.



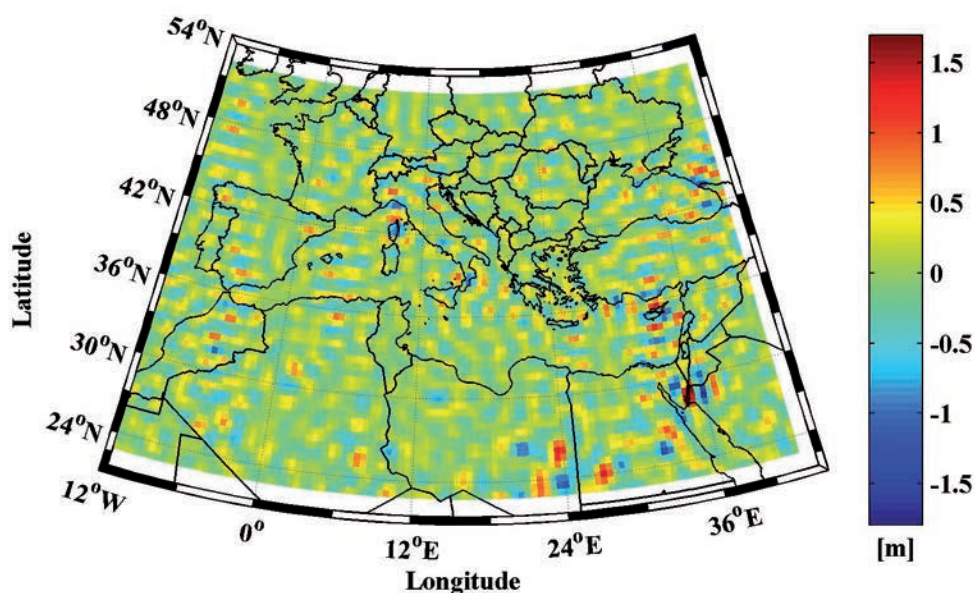


Fig. 6 - Difference between the geoid obtained synthesizing the GOCE and the EGM2008 model till degree 240.

Before concluding some general considerations are worth. First of all it is not possible to a priori say which of the different solutions is the best. The knowledge of the error covariance structure of EGM2008 spherical harmonic coefficients is in fact not complete (neither spectrally nor geographically) and, since an approximation has to be set up, we can never state that a more complicated approach necessarily brings to a better result. Therefore the selection of the best

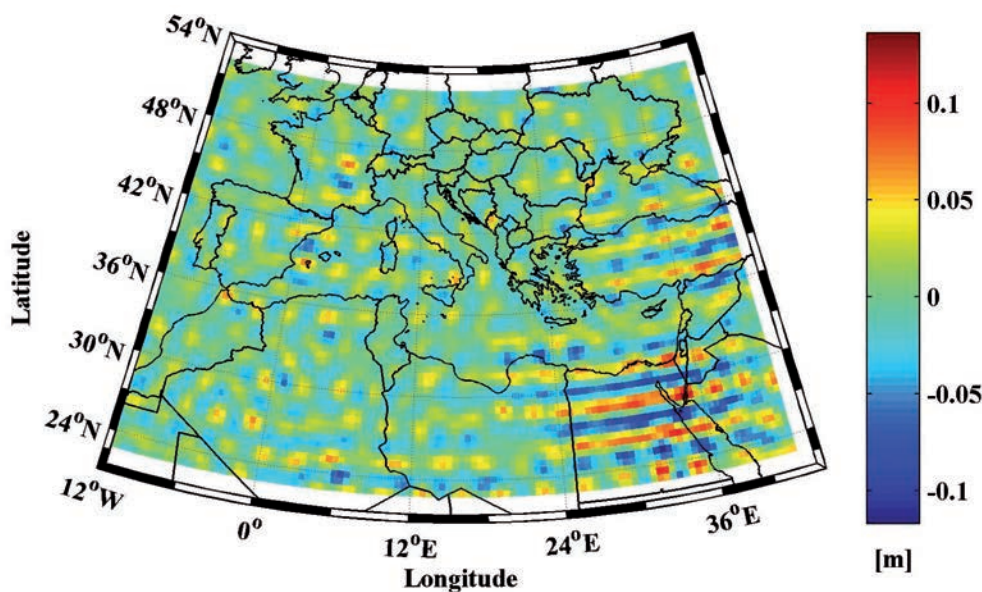


Fig. 7 - Difference between the geoid obtained synthesizing the merged model with block covariances (EGM08\_GOCE\_BC) and the one obtained considering coefficient variances (EGM08\_GOCE\_CV).

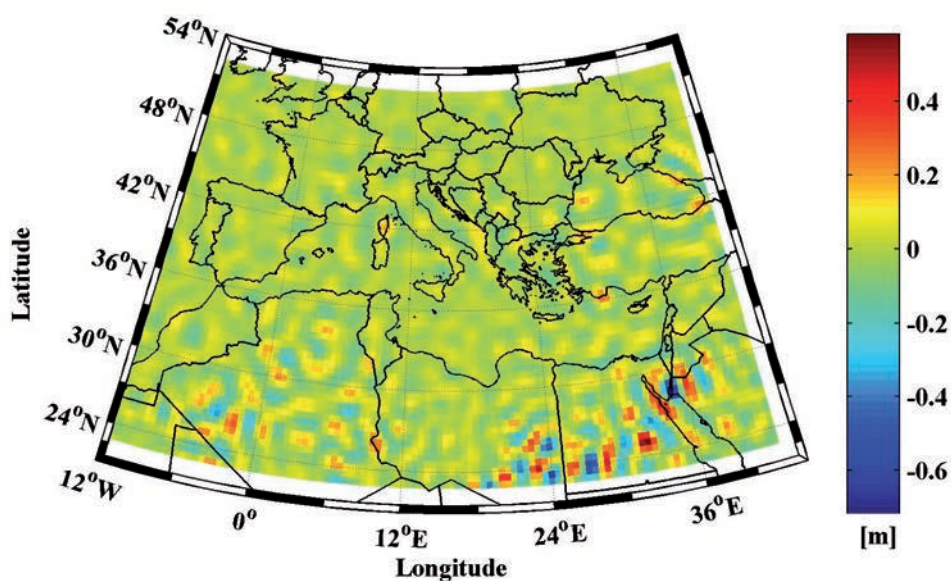


Fig. 8 - Difference between the geoid obtained synthesizing the locally adapted merged model (EGM08\_GOCE\_BC\_MED) and the one obtained using block covariances (EGM08\_GOCE\_BC).

merged model has to be based on an independent data set and can vary from area to area. In other words, a solution can be better than another over the Mediterranean Sea, but this does not imply that it is the best one in any region of the world. In particular, according to the available drifter data, it comes out that for the Mediterranean Sea the best solution is EGM08\_GOCE\_BC (see Menna *et al.*, 2012).

Note that, if one starts the combination from the original EGM2008 observations and not from the spherical harmonic coefficients, then the assumption of knowing the error covariance of these observations is more reasonable and therefore the quality of the results can be better controlled. As a matter of fact, the closer we are to the original observations the better is the description of their covariance structure.

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## Appendix

We want to prove the statement of the Remark of Section 3.

So we assume we have a 2D vector  $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  and two observations:

$$\begin{cases} Y = \underline{x} + \underline{\varepsilon} \\ Z = \underline{x} + \underline{\nu} \end{cases} \quad (\text{A1})$$

such that:

$$C_{YY} = C_{\varepsilon\varepsilon} = K = I + R, \quad R = \begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix}, \quad |\rho| \leq 1 \quad (\text{A2})$$

$$C_{ZZ} = C_{\nu\nu} = C = \frac{1}{2} \begin{bmatrix} 1 + \lambda & 1 - \lambda \\ 1 - \lambda & 1 + \lambda \end{bmatrix}, \quad \lambda > 0. \quad (\text{A3})$$

We note that:

$$\sigma^2(Y_1) = 1, \quad \sigma^2(Z_1) = \frac{1 + \lambda}{2} \quad (\text{A4})$$

so that the weighted average estimator of  $x_1$  (which does not imply any knowledge of  $\rho$ ) is:

$$\tilde{x}_1 = \frac{(1 + \lambda)Y_1 + 2Z_1}{3 + \lambda}. \quad (\text{A5})$$

Its variance is given by:

$$\sigma^2(\tilde{x}_1) = \frac{(\lambda + 1)}{(\lambda + 3)}. \quad (\text{A6})$$

We also note that  $\sigma^2(\tilde{x}_1) < 1$  always, and if we let  $\lambda \rightarrow \infty$ , we find  $\sigma^2(\tilde{x}_1) \rightarrow 1$  as it should be.

Now we try to estimate  $\tilde{\underline{x}}$  by ignoring that  $C_{\varepsilon\varepsilon} = I + R$ , knowing only the diagonal of  $C_{\varepsilon\varepsilon}$ , namely  $I$ . In this case the combined solution reads:

$$\tilde{\underline{x}} = (I + C^{-1})^{-1}(\underline{Y} + C^{-1}\underline{Z}); \tag{A7}$$

we observe that indeed  $\tilde{\underline{x}}$  is an unbiased estimator of  $\underline{x}$  as far as  $E\{\underline{\varepsilon}\} = E\{\underline{\nu}\} = 0$ . Now we use Eq. (22) to find the true covariance of  $\tilde{\underline{x}}$ , namely:

$$C_{\tilde{\underline{x}}\tilde{\underline{x}}} = (I + C^{-1})^{-1} + (I + C^{-1})^{-1}R(I + C^{-1})^{-1}. \tag{A8}$$

A direct computation shows that for the first component of  $\tilde{\underline{x}}$ , one has:

$$\sigma^2(\tilde{x}_1) = \frac{1 - 3\lambda}{4(1 + \lambda)} + \frac{\rho(1 + 3\lambda)(1 - \lambda)}{8(1 + \lambda)^2}. \tag{A9}$$

Now we see that:

$$\lim_{\substack{\lambda \rightarrow +\infty \\ \rho \rightarrow -1}} \sigma^2(\tilde{x}_1) = \frac{3}{4} + \frac{3}{8} = \frac{9}{8} > 1. \tag{A10}$$

It follows that there are large positive values of  $\lambda$  and negative values of  $\rho$  values to  $-1$  such that:

$$\sigma^2(\tilde{x}_1) > \sigma^2(\tilde{x}_1). \tag{A11}$$

Our statement is proven.

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