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The memory damped seismograph

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ABSTRACT This study aims to discuss and quantify the response and the decaying oscillations of a seismograph. The assumption is that instrument recording of the signal is governed by a second order differential equation including a memory formalism. We see that, in general, the response after an impulse is formed by a decaying step variation followed by oscillations with decreasing amplitude. In fact, the singularities of the solutions of the equation in the Laplace transform domain imply the periodic part of the Green function and the decay. We show how the memory stabilizes the system. It is seen theoretically that the classic mathematical model of the seismometer is not very suitable to model the modern version of the seismometer and that the memory model is possibly more appropriate.

Key words: memory, dynamics, response curve, seismograph.

1. Introduction

The knowledge of earthquakes and of the interior of the Earth is mostly based on the instruments called seismometers which record the ground motions due to earthquakes or the explosions made to generate the elastic waves which explore the interior of the Earth. For an early history of these instruments see e.g., Giugliano (1983).

We have two basically different instruments to record the ground motions. The first is based on the principle of inertia, such as the pendulums of different types, called pendulum seismometers, the others are the instruments monitoring the variations of length of a very limited portion of the surface of the Earth called strain-meters. In both cases, instruments cover a very large frequency band ranging from those of the free modes of oscillation of the Earth to frequencies of 100 Hz; in this note we are concerned with the pendulum seismometers.

The concept on which those instruments are based is generally that of the harmonic oscillator appropriately damped and, since the instruments are generally located on the rotating Earth, the reference frame is not inertial.

In recent decades, we have seen a marvelous evolution of those instruments which have reached a great sensitivity in monitoring the motions of the Earth surface at locations carefully selected in order to avoid civilization noise or that caused by other natural phenomena such as sea waves or winds.

In most books the damped harmonic oscillator is mathematically modeled with a simple second order differential equation with the concrete advantage that the quantitative characteristics of the instrument are described by the values of very few parameters which allow a direct, and without intermediary, quantitative comparison with the properties of other similar instruments.

Since the time of the first seismograph, many important changes have been applied to the instrument; the changes are of mechanic and/or electronic type in order to obtain a response curve as flat as possible. These changes, especially those of general hysteretic type, are not accommodated in the classic linear second order differential equation assumed to model the seismographs.

The present note is a simple contribution to the previous theoretical work consisting in the introduction of the mathematical memory operator, represented by the fractional order derivative, in the classic oscillator equation in order to model realistically different types of instruments. The equation used here to model the seismometer may be rightly defined as phenomenological. The same equation has been successfully used also to model the cycles of markets and population (Caputo, 2013).

The reputation of this type of equations, as stated in recent motivations for assigning Nobel prizes for physics, has been rehabilitated for their important contribution given in various forms to the rapid developments of the studies on the superconductive materials.

The applications of mathematical memory formalism is spread in many disciplines which directly concern our society and, to my knowledge, I cautiously say that, in most cases, in recent studies the tool of the mathematical memory formalism called fractional derivative was used.

There is some similarity between memory formalisms and hysteresis. However, the latter is a typical phenomenon of materials with different origin in different materials and also with effects on the material depending on the cause generating it. The word hysteresis is also accepted in fields which would be difficult to accept as *materials* such as economy, finance, social studies, psychology.

The mathematical memory formalism instead is an abstract tool which may be used to model many different phenomena in which the status of the system depends on its past. In the present note we replace the first order derivative of the equation of the classic seismometer with the mathematical memory operator represented by the fractional order derivative, in order to realistically represent the response curve of more complex instruments.

A study of the memory-damped harmonic oscillator may be done also substituting the classic first order derivative damping term with a fractional derivative of distributed order introduced by Caputo (1967). The simpler derivative of fractional order used here instead of that of distributed order, is less comprehensive but simpler and may allow a more direct a connection with the parameters used to describe the mechanical properties of the instrument. The damping of a mechanical system has been discussed also by Gaul *et al*. (1989) who used Fourier Transform (FT) to find the impulse response.

2. The memory equation

Let us consider the classic second order differential equation of the damped oscillator:

$$y''(t) + dy'(t) + gy(t) = f(t) \qquad y(0) = 0, \quad y'(0) = 0 \tag{1}$$

where $f(t)$ is a forcing function, the term $dy'(t)$ represents a damping or the damping is proportional to the first order derivative, g approximates the square of the frequency of free oscillation of the instrument and $h = d / 2\sqrt{g}$ is the damping constant.

If one wants to generalize Eq. (1) in order to represent the response curve and the damping of more complex instrument, a possibility is to substitute the first order derivative of Eq. (1) with the memory formalism of the derivative of real order.

In order to introduce the memory, the classic model Eq. (1) is replaced with:

$$y''(t) + \alpha D^{(v)} y(t) + \beta y(t) + r(t) = 0 \qquad (2)$$

where $v = m/n$ $(0 < m < n)$, $r(t)$ represents the perturbation and $D^{(v)}$ is the operator:

$$D^{(v)} f(t) = \{1/[\Gamma(1-\nu)]\} \int_0^t [df(\tau) / d\tau]\, d\tau / (t-\tau)^{(v)} \qquad (3)$$

representing the damping. The operator $D^{(v)}$ (Caputo, 1967) is called, perhaps improperly, fractional derivative of order $v \in [0,1]$; it has been thoroughly studied by many authors (e.g., Kiryakova, 1994; Podlubny, 1999; Kilbas and Marzan, 2006; Kilbas *et al.*, 2006; Diethelm, 2010; Mainardi, 2011). Jiao *et al.* (2012) made extensive and interesting applications of the fractional derivative of distributed order introduced by Caputo (1967).

In the classic case, represented by Eq. (1), in order to obtain decaying exponential solutions one selects the parameters α and β with the conditions $\alpha < 0$, $\alpha^2 < 4\beta$. In our case, the classic equation is generalized also in the sense that we consider α and β generic parameters only indirectly related to properties of the instrument. The relation to the instrumental properties, such as the proper period, are actually found in the poles of the solutions of the equation which, in turn, depends on the parameters α, β, and v. In other words, Eq. (2) is a phenomenological equation and the instrument represented should be more appropriately called pseudo-oscillator.

The fractional derivative is a frequently used tool in applied science and has been applied in many field such as theoretical physics (Naber, 2004), mathematics (Kilbas and Marzan, 2006) and biology (Caputo and Cametti, 2004); for a tentative list of these fields see e.g., Caputo (2013).

3. The stabilizing effect of the memory

In order to solve Eq. (3) we take its Laplace Transform (LT) with p LT variable and assume that:

$$y(0) = 0, \quad y'(0) = y_0$$

$$p^2 Y - y_0 + \alpha p^v Y + \beta Y + LT[r(t)] = 0 \qquad (4)$$

$$Y = \{- LT[r(t)] + y_0\} / (p^2 + \alpha p^v + \beta).$$

The case when the perturbation is represented by a power law is of interest to study the stabilizing effect of the memory. We then assume that $r(t) = g\, t^w$ and find:

$$LT[r(t)] = R(p) = g[\Gamma(1 + w)]p^{-w-1}. \quad (5)$$

Substituting in Eq. (4) we find:

$$Y = \{-g[\Gamma(1 + w)]p^{-w-1} + y_0\} / (p^2 + \alpha\, p^v + \beta). \quad (6)$$

The asymptotic value of $y(t)$ is then readily found with the use of the Extreme Value Theorem (EVT) (e.g., McCollum and Brown, 1965):

$$\lim_{p\to 0} pY(p) = \lim_{p\to 0} [-\, g\Gamma(1 + w)\, p^{-w)}] / \beta = \infty, \text{ when } w > 0$$
$$\lim_{p\to 0} pY(p) = \lim_{p\to 0} [-\, g\Gamma(1 + w)\, p^{-w)}] / \beta = -\, g / \beta, \text{ when } w = 0 \quad (7)$$
$$\lim_{p\to 0} pY(p) = \lim_{p\to 0} [-\, g\Gamma(1 + w)\, p^{-w)}] / \beta = 0, \text{ when } w < 0.$$

Which imply that the memory has a stabilizing effect if the exponent of the power law is nil, implying a constant perturbation, or when $w < 0$, implying a perturbation monotonically decreasing to zero. In both cases the effect of the memory will eventually absorb the perturbation. In the case when the perturbation is increasing ($w > 0$) the system will lead to an extreme condition.

4. The effect of the perturbation

The LT of the effect of the perturbation $r(t)$ assuming $y(0) = y'(0) = 0$ is:

$$Y_r(p) = [-\, LTr(t)] / (p^2 + \alpha p^v + \beta). \quad (8)$$

In order to find the effect $Y_r(t)$ of the perturbation we begin setting:

$$G(t) = LT^{1}\, [1 / (p^2 + \alpha p^v + \beta)] \quad (9)$$

and write:

$$Y_r(t) = -[r(t)]{*}G(t). \quad (10)$$

The LT of Eq. (9) is obtained by means of integration on the Bromwich-Hankel path of the complex plane finding:

$$G(t) = [\alpha\beta\, (\sin \pi v) / \pi]$$
$$\int_0^\infty [\exp(-ut)]u^v\, du / [(u^2 + \beta)^2 + 2\alpha u^v\, (u^2 + \beta) \cos(\pi v) + \alpha^2\, u^{2v}] + \quad (11)$$
$$+ \exp(at) / (2a + v\alpha\, a^{v-1}) + \exp(bt) / (2b + v\alpha\, b^{v-1)})$$

where a and b are the poles of Eq. (9) and the last line of Eq. (11) is the sum of the residues of Eq. (9) computed with the classic method. In other words, $G(s)$ represents the response to an impulse.

We note that, the poles of Eq. (9) are complex conjugate, the residues are also complex conjugate and their sum is real formed by the product of a sinusoidal wave with an exponential which has a negative exponent since α, β and v are generally selected to that purpose; moreover, the integral appearing in Eq. (11) is a monotonically decaying function of t which we call *transient*. The Eq. (11) is then formed by the sum of two decaying terms as in the classic case.

With the assumption that $\{-\ TL[r(t)]\}$ has no poles which, unless specifically expressed, is verified in all the cases examined here, taking into account Eq. (10), we finally obtain:

$$Y_r(t) = [\alpha(\sin \pi v) / \pi] \int_0^\infty [-r(t)]*[\exp(-ut)]\, u^v\, du /$$

$$[(u^2+\beta)^2 + 2\alpha u^v (u^2+\beta)\cos(\pi v) + \alpha^2 u^{2v}] + \qquad (12)$$

$$-[r(t)]*[\exp(at) / (2a + v\alpha\, a^{v-1}) + \exp(bt) / (2b + v\alpha\, b^{v-1}))].$$

We note in the Fig. 1 the large (by a factor less that 0.01) reduction of the response output at 1 s. Also is clear that the transient is smaller for smaller values of v.

5. The effect of perturbations acting for a time bounded interval

In the applications, is of interest the case of a perturbation bounded in amplitude and duration to see that the memory will stabilize the system.

Let the perturbation be:

$$[1\text{-}H(t\text{-}d)]\, r(t) \qquad (13)$$

where $r(t)$ is bounded and defined in the interval $[0, d]$ and $H(t\text{-}d)$ is the unit step function and $t = d$. The LT of the Eq. (13) is:

$$LT[1\text{-}H(t\text{-}d)]\, r(t) = \int_0^d \exp(-pt) r(t) dt = R_d(p)$$

where $R_d(p)$ is bounded. *S*ubstituting in Eq. (10) we find:

$$Y_r(p) = -\, R_d(p) / (p^2 + \alpha p^v + \beta)$$

and, with the EVT, we obtain:

$$\lim_{p\to 0} [pY_r(p)] = \lim_{p\to 0} pR_p(p) / (p^2 + \alpha\, p^v + \beta) = r(\infty) \lim_{p\to 0} 1 / (p^2 + \alpha\, p^v + \beta) = 0 \qquad (14)$$

which shows that the effect of a perturbation bounded in amplitude and duration will be asymptotically nil. We note that this property is independent from the order of fractional differentiation including the case of the classic model.

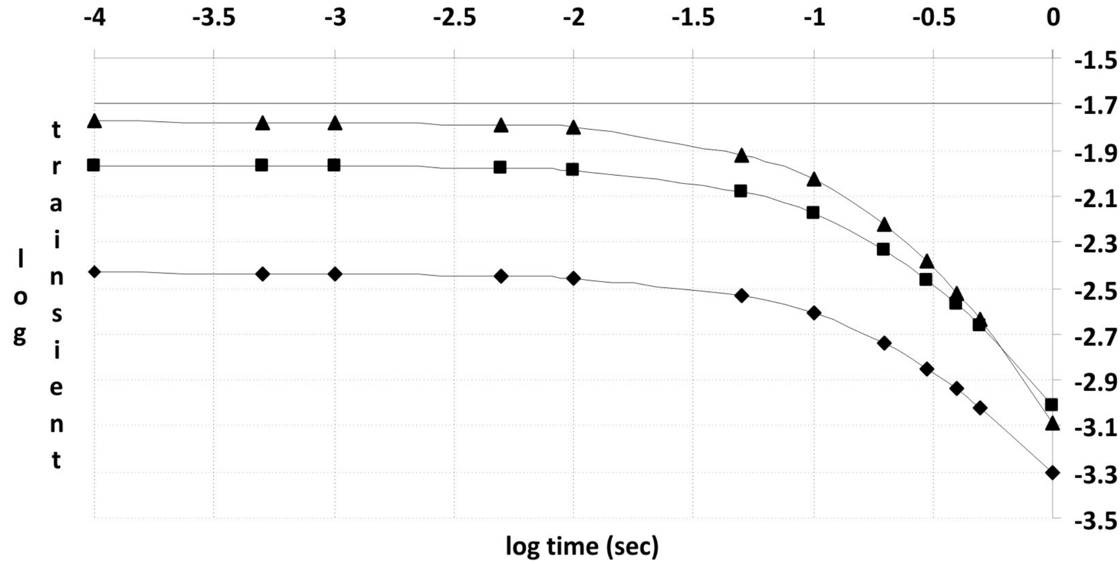

Fig. 1 - The transient of the memory seismograph with $\alpha = 2\pi$ and $\beta = (2\pi)^2$ for orders of differentiation: $v = 0.8$ (triangles), $v = 0.5$ (squares), $v = 0.2$ (diamonds). In the abscissa is the log of time (seconds).

6. The response to persistent perturbations

In the applications, is of interest the response of the system to a step function. In this case of a constant perturbation [$w = 0$ in Eq. (5)] of amplitude h, the Eq. (8) has an additional pole in the origin $p = 0$ and we find:

$$Y_h = h / p(p^2 + \alpha p^v + \beta) \tag{15}$$

Integrating Eq. (15) around the Bromwich-Hankel path of the complex plane, excluding the portion around the origin $p = 0$. Taking into account that there is a pole in $p = 0$ we find:

$$[h\alpha(\sin\pi v) / \pi] \int_0^\infty [\exp(-ut)] u^{v-1} du \, /$$
$$[(u^2 + \beta)^2 + 2\alpha u^v (u^2 + \beta)\cos(\pi v) + \alpha^2 u^{2v}] + h / \beta + \tag{16}$$
$$+ h \exp(at) / a(2a + \alpha v a^{v-1}) + h \exp(bt) / b(2b + \alpha v b^{v-1})$$

where:

$$h / \beta + h \exp(at) / a(2a + \alpha v a^{v-1}) + h \exp(bt) / b(2b + \alpha v b^{v-1})$$

is the sum of the residues of the poles in $p = 0$, $p = a$ and $p = b$ found with the classic method.

The poles a and b are complex conjugate, e.g., $a = m + if$, $b = m - if$, obtained solving the equation $p^2 + \alpha p^v + \beta = 0$. For stability we accept only values which generate converging solutions; then m must be negative.

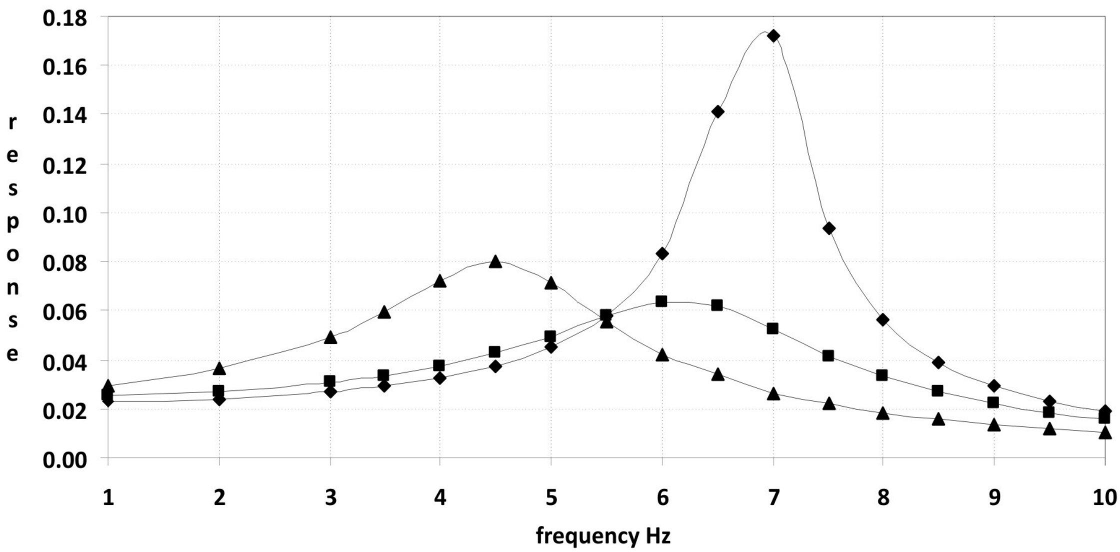

Fig. 2 - The response curves of the memory seismograph with $\alpha = 2\pi$ and $\beta = (2\pi)^2$ for orders of differentiation: $v = 0.8$ (triangles), $v = 0.5$ (squares), $v = 0.2$ (diamonds). In the abscissa is the frequency in Hz.

It is also seen that, since the contributions of the residues resulting from the poles in *a* and *b* is asymptotically nil, follows form Eq. (16) that the initial position of the sensor is asymptotically displaced by $y(\infty) = h/\beta$.

The integral in Eq. (16), generally decreasing more rapidly than the residues, represents what we call here the *transient* part of the response of the instrument.

Is of interest in applications the response to a periodic perturbation $r(t) = \sin wt$; it is obtained from Eq. (12) by direct substitution:

$$Y_r(t) = - [\alpha(\sin\pi v) / \pi] \int_0^\infty \{(\sin wt)*[\exp(-ut)]\} u^v \, du \, /$$

$$[(u^2 + \beta)^2 + 2\alpha u^v (u^2 + \beta) \cos \pi v + \alpha^2 u^{2v}] + \qquad (17)$$

$$- \sin wt*[\exp(at) / (2a + v\alpha\, a^{v-1)}) + \exp(bt) / (2b + v\alpha b^{v-1)})]$$

or more explicitly:

$$Y_r(t) = - [\alpha(\sin\pi v) / \pi] \int_0^\infty - \{[(-u \sin wt - w\cos wt) + w \exp(-ut)] / (w^2 + u^2)\} u^v \, du \, /$$

$$[(u^2 + \beta)^2 + 2\alpha u^v (u^2 + \beta) \cos(\pi v) + \alpha^2 u^{2v}] + \qquad (18)$$

$$- [(a \sin wt - w \cos wt) + w \exp(at)] / (w^2 + a^2) (2a + v\alpha a^{v-1)}) +$$

$$-[(b \sin wt - w \cos wt) + f \exp(bt)] / (w^2 + b^2) (2b + v\alpha b^{v-1)}).$$

It is readily seen that the response consists of a *transient* resulting from the integral plus a wave decaying at an exponential rate given by the real part *m* of the poles $a = m + if$, $b = m - if$ of Eq. (9) and also the persisting wave, with frequency *w*, modified in amplitude and phase.

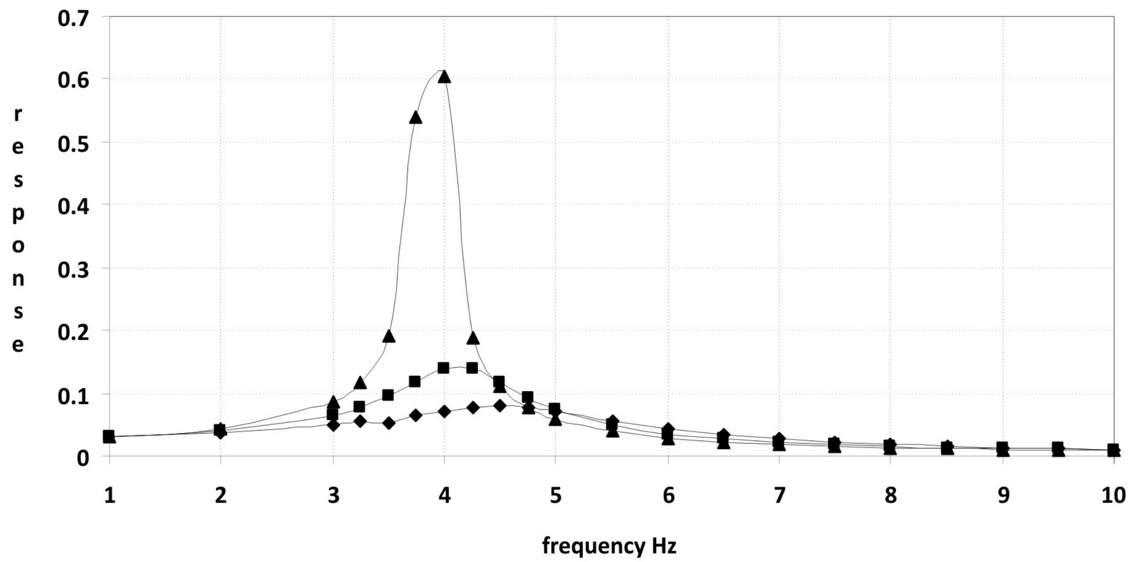

Fig. 3 - The response curves of the memory seismograph with $\alpha = 2\pi$ and $\beta = (2\pi)^2$ for orders of differentiation: $v = 0.8$ (diamonds), $v = 0.9$ (squares), $v = 1$ (triangles). In the abscissa is the frequency in Hz.

There are similarities with the classic model. Also in the classic equation modelling the seismometer the damping affects the free modes of the instrument and causes a constant change in the phase and the amplitude of the incoming sinusoidal wave. A difference is the presence of the transient represented by the rapidly decaying integral in the memory model.

The more important difference, however, is that the dynamic of the classic model is defined by two parameters: one representing the modes of the instrument, the other defining the damping. The dynamic of the memory model, particularly the damping, is concurrently represented by the three parameters α, β and v defining the poles and the residues, which in turn allow modelling a broad variety of complex instruments implying their dynamics.

7. The quality factor 1/*Q*

The mathematical expression of $1/Q$ is readily obtained from the real part m of the poles of Eq. (4).

To this purpose, as we noted already, the sinusoidal term of the response is given by the sum of the residues of the complex conjugate poles of Eq. (9) whose amplitude is governed by a negative exponential as in the classic model. If $m < 0$ is the real part of the poles the $1/Q$ is:

$$1 / Q = [1 - \exp (4m\pi / f)] / 2\pi \tag{19}$$

where f is frequency and $m < 0$. If $\alpha = 0$ we obtain $m = 0$ and we have the case of the harmonic oscillator without damping, if $v = 1$ we have the case of the classic damped oscillator with $m = -\alpha/2$. With the values of α and β used for the Figs. 1, 2 and 3 it is readily seen that the values of $1/Q$ are decreasing with increasing v.

8. The response curve of the memory model

The response curve of the memory model is obtained from the LT of Eq. (9) with the substitution $p = i\omega$, it is:

$$1 / [(-\omega^2 + \alpha\omega^\nu \cos \pi \nu + \beta)^2 + (\alpha\omega^\nu \sin \pi \nu)^2]^{0.5}. \quad (20)$$

The response of the memory model represented in Figs. 2 and 3 for different values of order of fractional differentiation shows that the peaks of the curves have a larger value for $\nu = 1$ and that the frequency of the maxima of the curves decreases with decreasing order of differentiation. In Fig. 3 it appears also that the curve with broader range of almost constant response would be that with order of differentiation $\nu = 0.5$ whose peak is 10 time smaller than that for $\nu = 1$.

Fig. 3 shows the confrontation of the response curves for relatively larger order of fractional differentiation including that of order $\nu = 1$ (the classic model of the seismograph) and those with order near 1: $\nu = 0.8$ (diamonds), $\nu = 0.9$ (squares), $\nu = 1$ (triangles). One may note that the curve of the classic model of the seismograph has a much larger peak with steeper variation around its maximum while the curves with smaller order of fractional differentiation have a smoother variation and, most important, a smaller peak amplitude.

9. Discussion and conclusions

It is generally assumed that the introduction of the first order derivative in the equation of a harmonic oscillator be an appropriate form to represent the damping properties of oscillators. However, one must consider that in many instruments, such as the seismometers, the damping is caused by a combination of mechanical and electric components. The damping is then due to electrical and mechanical causes.

Cole and Cole (1940) have shown experimentally that the complex dielectric constant is expressed by the formula:

$$d = \{[\varepsilon_s + \varepsilon_\infty (i\omega\tau)^\nu] / [1 + (i\omega\tau)^\nu]\}\, e \quad (21)$$

where e and d are the LT of the applied filed and of the displacement respectively, τ is a relaxation time, ω frequency, ε_s and ε_∞ are the static and infinite frequency dielectric constants and ν is a real number. Using the LT theorem of the fractional derivatives with $0 < \nu < 1$ (e.g, Caputo, 1967; Podlubny, 1999):

$$LTD^{(\nu)} f(t) = s^\nu F(s) - s^{\nu-1} f(0) \quad (22)$$

where s is the LT variable, is readily seen that the time domain form of the relation between applied the electric field and the displacement is expressed by means of derivatives of fractional order ν.

Recent studies of the anelastic materials have shown that (Caputo, 1967; Bagley and Torvik, 1985) a more appropriate mean to represent the properties of anelastic materials is to introduce the derivative of fractional order in the stress strain relations.

Since the damping of modern seismometers is based on anelastic and dielectric material, it seems reasonable to represent their damping properties by means of fractional derivative of appropriate order as is done in Eq. (2). The response curve of the seismograph is then governed by 2 parameters: the order of the fractional derivative and the factor in front of it.

It is seen in this note that, in the case of the model with the fractional derivative, the response to an impulse is formed by a rapidly decreasing transient and an exponentially decaying sinusoid. The type of transient represents the most apparent difference with the classic model. However, the most significant difference is in the response curve which may be designed to have a smooth variation around its peak. Both models have the same type of response to a periodic signal which in both cases is made of a non-decaying term and an exponentially decaying periodic output with modified phases and amplitudes.

The quality factor of the response curve of the memory model seismograph considered here depends on more than one parameter and, in some cases, could be close to frequency independence, for instance when the order of fractional differentiation is near 0.5, which of interest in the frequent case of broad band instruments.

Acknowledgements. I wish to thank Rodolfo Console for reading the paper and giving suggestions which made this note clearer and richer.

Appendix A. The LT^{-1} *G*(*p*)

In order to find the LT^{-1} $G(p)$ defined by Eq. (4) we set:

$$p = r\exp(\grave{\imath}\phi) \tag{A1}$$

and integrate on the Bromwich-Hankel path in the complex plane finding:

$$[1/(2\pi\, i)] \int_0^\infty [\exp(-ut)]\, du\, \{1/[u^2 + \alpha\, u^\nu \exp(-i\pi\, \nu) + \beta] - 1/[u^2 + \alpha\, u^\nu \exp(i\pi\, \nu) + \beta]\} =$$

$$= [\alpha(\sin \pi\, \nu) / \pi] \int_0^\infty [\exp(-ut)]\, u^\nu\, du\, /$$

$$[u^4 + \alpha\, u^{2+\nu} \exp(-i\pi\, \nu) + u^2 \beta + \alpha\, u^{2+\nu} \exp(i\pi\, \nu) + \alpha^2 u^{2\nu} + \alpha\beta u^\nu \exp(i\pi\, \nu) +$$

$$+ \beta u^2 + \alpha\beta u^\nu \exp(-i\pi\, \nu) + \beta^2]$$

and finally:

$$G(t) = [\alpha\beta\, (\sin \pi\, \nu) / \pi] \int_0^\infty [\exp(-ut)]\, u^\nu\, du\, / [(u^2 + \beta)^2 + 2\alpha u^\nu\, (u^2 + \beta) \cos(\pi\, \nu) + \alpha^2 u^{2\nu}]. \tag{A2}$$

Appendix B. The LT^{-1} *Y*$_r$(*p*)

In order to find the LT^{-1} of $Y_r(p)$ defined by Eq. (15), that is, when the input is a step function, we set:

$$p = r\exp(\grave{\imath}\phi) \tag{B1}$$

and, as it was done in Appendix A, we integrate on the Bromwich-Hankel path in the complex plane taking into account that there is a pole in $p = 0$, which implies that the path must go around the origin. We find:

$$[1 / (2\pi\, i)] \int_0^\infty [\exp(-ut)]\, (du / u)\, \{1 / [u^2 + \alpha u^\nu \exp(-i\pi\, \nu) + \beta] + - 1 / [u^2 + \alpha u^\nu \exp(i\pi\, \nu) + \beta]\} + h / \beta \tag{B2}$$

or:

$$[h\alpha\, (\sin \pi\, \nu) / \pi] \int_0^\infty [\exp(-ut)] u^{\nu-1}\, du\, / [(u^2 + \beta)^2 + 2\alpha u^\nu\, (u^2 + \beta) \cos(\pi\, \nu) + \alpha^2 u^{2\nu}] + h / \beta \tag{B3}$$

where h / β is the contribution of the integration around $p = 0$.

REFERENCES

Bagley R.L. and Torvik P.J.; 1985: *Fractional calculus in the transient analysis of viscoelastically damped structures*. AIAA J., **23**, 918-925.

Caputo M.; 1967: *Linear model of dissipation whose Q is almost frequency independent - II*. Geophys. J. R. Astron. Soc., **13**, 529-539.

Caputo M.; 2013: *Memory oscillator for modeling socio-economic oscillatory responses to extreme events*. Economia Politica, **30**, 231-239.

Caputo M. and Cametti C.; 2004: *Memory formalism in the passive diffusion across a biological membrane*. J. Membr. Sci., **250**, 79-84.

Cole K.S. and Cole R.H.; 1940: *Dispersion and absorption in dielectrics - I Alternating current characteristic*. J. Chem. Phys., **9**, 341-351.

Diethelm K.; 2010: *The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type*. Lect. Notes Math., n. 2004, Springer-Verlag, Heidelberg, Germany, 255 pp.

Gaul L., Klein P. and Kemple S.; 1989: *Impulse response function of an oscillator with fractional derivative in damping descriptio*. Mech. Res. Appl., **16**, 297-305.

Giugliano M.; 1983: *Luigi Palmieri e i suoi apparecchi elettrici nella storia delle Scienze*. Annuario 1983, Edizioni A.S.M.V., Piedimonte Matese, Caserta, Italy, 24 pp.

Jiao Z., Chen Y. and Podlubny I.; 2012: *Distributed order dynamic systems, modeling, analysis and simulation*. Springer, New York, NY, USA, 90 pp.

Kilbas A.A. and Marzan S.A.; 2006: *Cauchy problem for differential equation with Caputo derivative*. Fractional Calculus & Appl. Anal., **7**, 296-321.

Kilbas A.A., Srivastava H.M. and Trujillo J.J.; 2006: *Theory and applications of fractional differential equations*. North-Holland Math. Stud., **204**, 523 pp.

Kiryakova V.S.; 1994: *Generalized fractional calculus and applications*. Longman Sci. and Tech., Harlow and Wiley Ltd., New York, NY, USA, 388 pp.

Mainardi F.; 2011: *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. Monograph, Imperial College Press, London, UK, 340 pp.

McCollum P.A. and Brown B.F.; 1965: *Laplace transform tables and theorems*. Holth, Rinehart and Winston, New York, NY, USA, 117 pp.

Naber M.; 2004: *Time fractional Schrödinger equation*. J. Math. Phys., **45**, 3339-3352.

Podlubny I.; 1999: *Fractional differential equations*. Academic Press, New York, NY, USA, 364 pp.

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