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On the boundary condition of the Sverdrup stream function

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ABSTRACT The classical problem for ascribing, on theoretical grounds, the longitudinal boundary condition to the Sverdrup transport is faced and solved by means of new proof.

It is important to keep in mind that the formulation of a problem in fluid mechanics requires the specification of the pertinent equations and the relevant boundary conditions to be considered with equal care and attention.

Joseph Pedlosky

Key words: Sverdrup balance, boundary conditions, sub-tropical and sub-polar gyres.

1. Introduction

The problem in ascribing the longitudinal boundary condition to the transport stream function of Sverdrup (1947) is a classic one, for historical reasons. In 1947 the Norwegian oceanographer Harald Ulrik Sverdrup was led to derive the relation, which now bears his name, Sverdrup balance, to explain the surface counter current of the Pacific Ocean at a latitude between 5° and 10° N, where it flows against the trade winds. To make his model, in agreement with observations, the author forced the zonal transport to vanish at the eastern oceanic boundary but he did not provide any explanation of his choice. However, in the framework of the Sverdrup balance (not in that of a full model) there is no way to justify the choice between the eastern or western boundaries where the zonal Sverdrupian transport vanishes. The balance is unable to satisfy the no mass-flux across both the meridional coasts and to guarantee mass conservation in the basin as a whole. To complete the dynamics, frictional dissipation must be introduced in the motion equation to bring circulation to a steady state. With the aid of this ingredient, a number of theoretical arguments in favour of the correct boundary condition of the Sverdrup flow have been produced, in particular those of Stommel (1965) and Pedlosky (1965), which are worthy of mention. Less convincing proofs are not cited in this paper.

Stommel (1965) provided a semi-qualitative explanation of the asymmetry of the flow pattern for a subtropical gyre, based on the fact that a frictional western boundary-layer is a source of counter-clockwise vorticity whereas a frictional eastern boundary layer would be a source of clockwise vorticity.

Pedlosky (1965) gave a fascinating and fully different approach to the same E-W asymmetry. He pointed out the different behaviour of long and short decaying Rossby waves while reflecting eastwards and westwards. As a consequence of this phenomenon, a western boundary layer occurs, while on the eastern region no boundary layer forms, thus allowing a Sverdrup regime extended to the eastern coastline. In both cases the boundary condition of the Sverdrup flow follows from the localisation of the return current.

The aim of this investigation is to infer the boundary condition of the Sverdrup transport stream function in the framework of the quasi-geostrophic dynamics, within the dissipative, linear and steady context while completely neglecting westward intensification. The quasi-geostrophic approach is the most natural in dealing with ocean circulation in which the horizontal length scale is large enough for β-plane approximation but smaller than the Earth's radius, as in the case of sub-tropical and sub-polar gyres. Dissipation is requested to assure the steadiness of the motion on time scales, which filter out the high frequency of the eddy variability. Ultimately, linearity of the governing vorticity equation enables the application of the superposition principle, which is an indispensable prerequisite to complete the deductive process. The flow field is given by the superposition of that Sverdrupian with another field aimed to close the overall transport steam lines. The inference is mainly based on an inequality originated by the sink nature, from the energetic point of view, of dissipation.

2. Governing equations and basic inequalities

A flow governed by the dimensionless steady quasi-geostrophic vorticity equation in the dissipative regime:

$$\frac{\partial \psi}{\partial x} = curl_z \boldsymbol{\tau} + \left(-1\right)^n \left(\delta_n / L\right)^{2n-1} \nabla^{2n} \psi \tag{1}$$

in a certain β– plane is the starting assumption. In Eq. (1), ψ (x, y) is the transport stream function and δ_n/L (<<1) is the frictional boundary-layer width relative to a certain parameterisation of turbulence, singled out by the integer, n (≥1). The Stommel and Munk classical models are easily recovered for n = 1 and n = 2, respectively. Due to the smallness of δ_n/L, the Sverdrup balance emerges from Eq. (1) wherever dissipation plays a minor role. In such case, the meridional transport, $\partial\psi/\partial x$, tends to adjust in balance with the wind-stress curl, according to the Sverdrup equation:

$$\frac{\partial \psi_{Sv}}{\partial x} = curl_z \boldsymbol{\tau} \tag{2}$$

where $curl_z\boldsymbol{\tau} = \hat{k}\cdot\nabla\times\boldsymbol{\tau}$. The fluid domain D is given by:

$$D = \left\{\left(x, y\right):\ x_W \le x \le x_E,\ 0 \le y \le 1\right\} \tag{3}$$

where x_W and x_E may depend on latitude. Latitudes y = 0 and y = 1 are, by definition, two consecutive zeros of $curl_z\tau$ while the sign of the latter establishes whether the gyre is sub-tropical ($curl_z\boldsymbol{\tau} \le 0$) or sub-polar ($curl_z\boldsymbol{\tau} \ge 0$). In both cases, the signs of $curl_z\boldsymbol{\tau}$, and hence of $\partial\psi_{Sv}/\partial x$, are constant throughout the gyre. The impermeability of the coastlines yields the kinematic boundary condition:

$$\psi\left(x, y\right) = 0 \quad \forall\left(x, y\right) \in \partial D. \tag{4}$$

Following these premises, the product of Eq. (1) multiplied by ψ and the subsequent integration on the domain (3) should be considered, with the aid of the Green theorem:

$$\frac{1}{2}\oint_{\partial D} \psi^2 dy - \int_D \psi \frac{\partial \psi_{Sv}}{\partial x} dxdy = (-1)^n (\delta_n / L)^{2n-2} \int_D \psi \nabla^{2n} \psi \, dxdy \, . \quad (5)$$

Due to the boundary condition (4), the first term on the left of Eq. (5) is zero. Moreover, in general:

$$(-1)^n (\delta_n / L)^{2n-2} \int_D \psi \, \nabla^{2n} \psi \;\, dx\,dy > 0. \quad (6)$$

Suitable additional boundary conditions are understood to carry out the integration appearing in inequality (6). The linearity of Eq. (1) enables the introduction of the quantity:

$$\phi = \psi - \psi_{Sv} \quad (7)$$

to meet the following requirements.

- No mass-flux across both the longitudinal boundaries, that is

$$\phi(x_W, y) + \psi_{Sv}(x_W, y) = 0 \quad (8)$$

and

$$\phi(x_E, y) + \psi_{Sv}(x_E, y) = 0 \, . \quad (9)$$

Only in one of Eqs. (8) and (9), $\phi \neq 0$; but in with of them is just the main question of the investigation.

- Mass conservation in the basin as a whole, that is:

$$\int_{x_W}^{x_E} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi_{Sv}}{\partial x} \right) dx = 0 \quad (10)$$

at every latitude.

The term $\phi\,(x, y)$ can be evaluated by means of standard boundary-layer techniques, however, only if the boundary condition of the Sverdrup stream function is known. The only question for which of the two boundary conditions, between $\psi_{Sv}\,(x_W, y) = 0$ and $\psi_{Sv}\,(x_E, y) = 0$, is that true?

To solve the problem, inequality:

$$\int_D \psi \frac{\partial \psi_{Sv}}{\partial x} dxdy < 0 \quad (11)$$

deriving from Eq. (5) and inequality Eq. (6) are taken into account. It is generated by the sink nature of dissipation, as anticipated in the introduction. Substitution of Eq. (7) into inequality (11) provides an inequality of the kind:

$$I_{Sv} + I_{\phi} < 0 \tag{12}$$

where

$$I_{Sv} = \int_D \psi_{Sv} \frac{\partial \psi_{Sv}}{\partial x} dxdy, \qquad I_{\phi} = \int_D \phi \frac{\partial \psi_{Sv}}{\partial x} dxdy . \tag{13}$$

Given that inequality (12) depends on ψ_{Sv} (x_E, y) and ψ_{Sv} (x_W, y), the same inequality can be symbolically written as:

$$I_{Sv}\left(\psi_{Sv}\left(x_E, y\right),\ \psi_{Sv}\left(x_W, y\right)\right) + I_{\phi}\left(\psi_{Sv}\left(x_E, y\right),\ \psi_{Sv}\left(x_W, y\right)\right) < 0. \tag{14}$$

Then, in the next section, it is shown that:

$$I_{Sv}\left(0,\ \psi_{Sv}\left(x_W, y\right)\right) + I_{\phi}\left(0,\ \psi_{Sv}\left(x_W, y\right)\right) < 0 \tag{15}$$

while

$$I_{Sv}\left(\psi_{Sv}\left(x_E, y\right),\ 0\right) + I_{\phi}\left(\psi_{Sv}\left(x_E, y\right),\ 0\right) > 0 . \tag{16}$$

The conclusion is that inequality (15) satisfies inequality (12) and yields the boundary condition:

$$\psi_{Sv}\left(x_E, y\right) = 0. \tag{17}$$

Conversely, inequality (16) contradicts inequality (12) and, therefore, the boundary condition ψ_{Sv} (x_W, y) = 0 is ruled out.

3. Proof of Eq. (17)

Integration of the quantities I_{Sv} and I_{ϕ} appearing in Eq. (13) is explicitly carried out here. The first of them is immediate, that is:

$$I_{Sv} = \frac{1}{2} \int_0^1 \left[\psi_{Sv}^2\left(x_E, y\right) - \psi_{Sv}^2\left(x_W, y\right)\right] dy. \tag{18}$$

An integration by parts of the second integral provides:

$$I_\phi = \int_0^1 \left\{ \left[\phi\psi_{Sv}\right]_{x_W}^{x_E} - \int_{x_W}^{x_E} \psi_{Sv} \frac{\partial \phi}{\partial x} dx \right\} dy \tag{19}$$

i.e.:

$$I_\phi = \int_0^1 \left[\phi\left(x_E, y\right)\psi_{Sv}\left(x_E, y\right) - \phi\left(x_W, y\right)\psi_{Sv}\left(x_W, y\right) - \int_{x_W}^{x_E} \psi_{Sv} \frac{\partial \phi}{\partial x} dx \right] dy. \tag{20}$$

Recalling boundary conditions in Eqs. (8) and (9), Eq. (20) becomes:

$$I_\phi = \int_0^1 \left[-\psi^{2\,Sv}\left(x_E, y\right) + \psi^{2\,Sv}\left(x_W, y\right) - \int_{x_W}^{x_E} \psi_{Sv} \frac{\partial \phi}{\partial x} dx \right] dy. \tag{21}$$

The meridional return flow, $\frac{\partial \phi}{\partial x}$, appearing in Eqs. (19) to (21), closes the circulation forced by the Sverdrup balance. It is opposite to the Sverdrup flow [Eq. (2)] and has a constant sign. Thus, according to the generalised mean value theorem:

$$\int_0^1 dy \int_{x_W}^{x_E} \psi_{Sv} \frac{\partial \phi}{\partial x} dx = \int_0^1 dy\, \psi_{Sv}\left(\tilde{x}, y\right) \int_{x_W}^{x_E} \frac{\partial \phi}{\partial x} dx. \tag{22}$$

where $x_W < \tilde{x} < x_E$. Moreover, owing to Eq. (10):

$$\int_0^1 dy\, \psi_{Sv}\left(\tilde{x}, y\right) \int_{x_W}^{x_E} \frac{\partial \phi}{\partial x} dx = \int_0^1 \psi_{Sv}\left(\tilde{x}, y\right)\left[\psi_{Sv}\left(x_W, y\right) - \psi_{Sv}\left(x_E, y\right)\right] dy. \tag{23}$$

The substitution of Eq. (23) into Eq. (21) results in the equation:

$$\begin{aligned} I_\phi &= \int_0^1 \left\{ -\psi_{Sv}^2\left(x_E, y\right) + \psi_{Sv}^2\left(x_W, y\right) - \psi_{Sv}\left(\tilde{x}, y\right)\left[\psi_{Sv}\left(x_W, y\right) - \psi_{Sv}\left(x_E, y\right)\right] \right\} dy \\ &= \int_0^1 \left\{ \psi_{Sv}\left(x_E, y\right)\left[\psi_{Sv}\left(\tilde{x}, y\right) - \psi_{Sv}\left(x_E, y\right)\right] + \psi_{Sv}\left(x_W, y\right)\left[\psi_{Sv}\left(x_W, y\right) - \psi_{Sv}\left(\tilde{x}, y\right)\right] \right\} dy. \end{aligned} \tag{24}$$

According to the Lagrange theorem:

$$\psi_{Sv}\left(\tilde{x}, y\right) - \psi_{Sv}\left(x_E, y\right) = \left(x_E - \tilde{x}\right) curl_z \boldsymbol{\tau}\left(\xi, y\right) \tag{25}$$

$$\psi_{Sv}\left(x_W, y\right) - \psi_{Sv}\left(\tilde{x}, y\right) = \left(\tilde{x} - x_W\right) curl_z \boldsymbol{\tau}\left(\lambda, y\right) \tag{26}$$

where $x_E - \tilde{x} > 0$, $\tilde{x} - x_W > 0$ and (ξ, y), (λ, y) are suitable points of the domain (3). Substitution of Eqs. (25) and (26) into Eq. (24) yields:

$$I_\phi = \int_0^1 \left[\left(x_E - \tilde{x}\right)\psi_{Sv}\left(x_E, y\right) curl_z \boldsymbol{\tau}\left(\xi, y\right) + \left(\tilde{x} - x_W\right)\psi_{Sv}\left(x_W, y\right) curl_z \boldsymbol{\tau}\left(\lambda, y\right)\right] dy. \qquad (27)$$

On the whole, from Eqs. (18) and (27), the following inequality is obtained:

$$I_{Sv} + I_\phi = \int_0^1 \Big[\tfrac{1}{2}\psi_{Sv}^2(x_E, y) - \tfrac{1}{2}\psi_{Sv}^2(x_W, y) + (x_E - \tilde{x})\psi_{Sv}(x_E, y) curl_z \boldsymbol{\tau}(\xi, y) + (\tilde{x} - x_W)\psi_{Sv}(x_W, y) curl_z \tau(\lambda, y)\Big] dy < 0. \qquad (28)$$

The latter is the explicit form of Eq. (14).

The alternative

a) $\psi_{Sv}(x_E, y) = 0$

b) $\psi_{Sv}(x_W, y) = 0$

under the hypotheses of a sub-tropical gyre and of a sub-polar gyre is, at this stage, taken into consideration.

- In case a), a sub-tropical gyre implies $\partial\psi_{Sv}/\partial x < 0$ and $\psi_{Sv}(x_W, y) > 0$, while a sub-polar gyre implies $\partial\psi_{Sv}/\partial x > 0$ and $\psi_{Sv}(x_W, y) < 0$. Thus, in both cases the product $\psi_{Sv}(x_W, y)\, curl_z \boldsymbol{\tau}(x, y) < 0$. Therefore, inequality (28) becomes

$$I_{Sv} + I_\phi = \int_0^1 \left[-\frac{1}{2}\psi_{Sv}^2\left(x_W, y\right) + \left(\tilde{x} - x_W\right)\psi_{Sv}\left(x_W, y\right) curl_z \boldsymbol{\tau}\left(\lambda, y\right)\right] dy < 0 \qquad (29)$$

i.e.:

$$I_{Sv} + I_\phi < 0 \qquad (30)$$

in accordance with inequality (15).

- In case b), a sub-tropical gyre implies $\partial\psi_{Sv}/\partial x < 0$ and $\psi_{Sv}(x_E, y) < 0$, while a sub-polar gyre implies $\partial\psi_{Sv}/\partial x > 0$ and $\psi_{Sv}(x_E, y) > 0$. Thus, in both cases the product $\psi_{Sv}(x_E, y)\, curl_z \boldsymbol{\tau}(x, y) > 0$. Therefore, inequality (28) becomes:

$$I_{Sv} + I_\phi = \int_0^1 \left[\frac{1}{2}\psi_{Sv}^2\left(x_E, y\right) + \left(x_E - \tilde{x}\right)\psi_{Sv}\left(x_E, y\right) curl_z \tau\left(\xi, y\right)\right] dy > 0 \qquad (31)$$

i.e.:

$$I_{Sv} + I_\phi > 0 \qquad (32)$$

in agreement with inequality (16), but in disagreement with inequality (12).

The conclusion is that only boundary condition a) satisfies inequality (28) and, therefore, the correct boundary condition to be applied to the Sverdrup balance is given by Eq. (17).

3.1. Remark 1

Eq. (11) provides the opportunity to return to counter-currents, which were the inspiring argument for the formulation of the Sverdrup balance. The left side of inequality (11) can be developed as follows:

$$\int_D \psi \frac{\partial \psi_{Sv}}{\partial x}\, dxdy = \int_D \psi\ \hat{\mathbf{k}}\cdot\nabla\times\boldsymbol{\tau}\ dxdy = \hat{\mathbf{k}}\cdot\int_D \left[\nabla\times\left(\psi\ \boldsymbol{\tau}\right) - \nabla\psi\times\boldsymbol{\tau}\right] dx\,dy\,. \tag{33}$$

By using the Stoke theorem and the geostrophic current, $\mathbf{u} = \hat{\mathbf{k}}\times\nabla\psi$, the term on the right of Eq. (33) becomes, with the aid of identity $\hat{\mathbf{k}}\cdot\nabla\psi\times\boldsymbol{\tau} = \boldsymbol{\tau}\cdot\hat{\mathbf{k}}\times\nabla\psi$,

$$\oint_{\partial D} \psi\ \boldsymbol{\tau}\cdot\hat{\mathbf{t}}\ ds - \boldsymbol{\tau}\cdot\hat{\mathbf{k}}\times\nabla\psi\ dxdy = -\int_D \boldsymbol{\tau}\cdot\mathbf{u}\ \ dx\,dy\,. \tag{34}$$

In Eq. (34), $\hat{\mathbf{t}}$ is the unit vector tangent along *∂D* and $\hat{\mathbf{k}}$ is the unit vector normal to the *β*-plane. Owing to the boundary condition of Eq. (4), the first term on the left of Eq. (34) is zero, whence inequality (11) and Eq. (34) imply:

$$\int_D \boldsymbol{\tau}\cdot\mathbf{u}\ \ dx\,dy > 0. \tag{35}$$

Inequality (35) shows that a counter-current, $\mathbf{u}_c$, which is characterised by $\boldsymbol{\tau}\cdot\mathbf{u}_c < 0$, is a statistically minority element, which does not invalidate inequality (35) itself, in the sense that the average $\left\langle\boldsymbol{\tau}\cdot\mathbf{u}\right\rangle = \frac{1}{D}\int_D \tau\cdot\mathbf{u}\ dxdy\ \left(>0\right)$ also includes $\mathbf{u}_c$, if any. This conclusion does not depend on the linearity of the vorticity equation: in fact, the non-linear acceleration term [absent in Eq. (1)] multiplied by the stream function and integrated on the fluid domain is exactly zero so inequality (35) is preserved in any case. Indeed, phenomenology exhibits counter-currents embedded in current fields whose characteristics reveal non-linear dynamics.

3.2. Remark 2

Crisciani and Purini (1997) considered the same problem as the one analysed in this investigation to obtain the boundary condition of Eq. (17). They used inequality (18) as well, but dealt more hastily with inequality (19), based only on scaling arguments. Thus, the derivation expounded in this study should be much more accurate and convincing.

4. Conclusions

Not surprisingly, the result of this investigation derives from the embedding of the Sverdrup balance into a vorticity equation with dissipation. The peculiarity of the inference lies in the development of inequality (11) determined only by the dissipation sink effect but without any hypothesis on the position of the frictional boundary layer. The system examined is largely idealised, however, the possible inclusion of further details (for instance bottom topography and coastline modulation) is expected not to modify the result but, rather, make it less evident.

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