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Ieley's regional model of wind-driven circulation: a tool to prove that intensification is westwards

F. Crisciani[1] and R. Mosetti[2]

[1] *Former Research Executive, CNR-ISMAR, Trieste, Italy*
[2] *Former Research Executive, Istituto Nazionale di Oceanografia e di Geofisica Sperimentale - OGS, Trieste, Italy*

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ABSTRACT Ierley's regional model of a single-gyre wind-driven circulation is an interesting compromise between the need to account for flow nonlinearity and its Sverdrupian behaviour far from the western area of the basin. The model assumes *a priori* that current intensification is westwards, but it can be easily reformulated, as the authors do, by setting the intensification region at the east. However, in this case the energetics of the flow exhibits an inconsistency that shows that westward intensification has no alternative. This is in accordance with all the previous conclusions on the subject but, for the first time, with the novelty of having taken into account also the nonlinear character of the flow outside the Sverdrupian area.

Key words: regional model, wind-driven circulation, westward intensification.

1. Introduction

Recently, Crisciani and Mosetti (2021) have revisited the physical foundations of westward intensification of wind-driven ocean currents in the framework of the quasi-geostrophic, linear dynamics of a single gyre. In the theory developed by the authors, westward intensification is the result of the superposition of two virtual components of the total transport stream function, one symmetric with respect to the mirror reflection at the central longitude of the basin, the other anti symmetric. The anti symmetric term exists if, and only if, the planetary vorticity gradient is taken into account by the dynamics; in this case, westward intensification is nothing but the local evidence of above superposition, that takes place in the full basin but it is hidden in the remainder of the gyre. The same paper reports also a short history of the attempts to explain physically this impressive phenomenon extended to all the ocean basins of the globe. At authors' knowledge, the number of papers devoted primarily to the cause of westward intensification, after Stommel (1948) and detached from any particular model, is relatively small (Pedlosky, 1965; Pond and Pickard, 1983; Cushman-Roisin, 1994; Crisciani and Purini, 1997; Crisciani and Mosetti, 2021). The researchers seem to have taken away interest on the topic. In particular, all the contributions known to the authors resort to linear models mainly because above a certain threshold of non linearity, the streamlines produced by numerical models exhibit a trend hardly compatible with that of the Sverdrup solution in the area, where the latter is expected to hold on the basis of observational evidence (see e.g. Welander, 1959; Wunsh, 2011). Boening's (1986) experiments are very explicative on this subject, in particular that shortly reported and commented by Pedlosky (1996) in section 2.12 of his textbook. For this reason, the noticeable regional model of Ierley

(1987) attracted the attention of the authors of this paper. In fact, Ierley assumes a net separation between the western region of the gyre, where non linearity and dissipation are active, from the remaining part, governed by the Sverdrup balance that gives the forcing term of the whole system. In the western region, intensification takes place but the effect of nonlinearity does not leave the region of intensification. In section 2, all the details will be highlighted.

Now, the basic idea of this investigation is the following: once Ierley's model is accepted, can it be reformulated setting the region of intensification at the east and the Sverdrupian part at the west of the latter? Section 3 proofs, *per absurdum* that this eventuality leads to a contradiction in the energy balance of the flow in the region of intensification, so intensification is necessarily westwards. Thus, Ierley's regional model gives a tool to prove westward intensification.

2. The regional model of Glenn R. Ierley

In the non dimensional model of Ierley (1987), a certain subtropical gyre is included into a rectangular domain of the beta plane, which splits longitudinally into two adjacent parts, one located at the westernmost area of the basin while the other extends from the eastern limit of the previous one up to the eastern boundary. The so-called regional model refers to the part at the west, where the steady circulation is governed by a quasi-geostrophic barotropic equation. Its solution is matched with that of Sverdrup extending from the eastern limit of above region up to the eastern boundary (see Fig. 1)

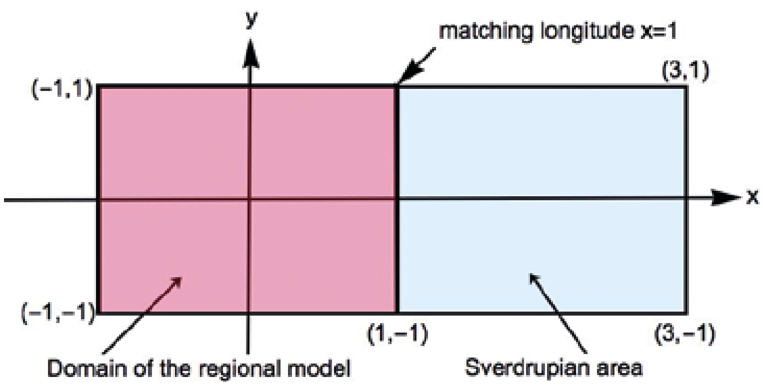

Fig. 1 - Sketch of the total fluid domain, with the specification of the region governed by Eq. 2 and the Sverdrupian area in which Eq. 15 holds.

This partition of the gyre assumes the westward intensification and its main purpose is to solve numerically the circulation model inside the region at the western area of the basin that must matched with the Sverdrup solution at the inner eastern limit of the western region. The model includes the dissipative mechanism consisting of lateral diffusion of relative vorticity while the forcing is expressed, in terms of a sinusoidal wind-stress curl, as boundary condition of the stream function at the inner eastern limit of the region.

The regional domain Q is defined as:

$$Q = \{(x, y): -1 \leq x \leq 1, -1 \leq y \leq 1\}. \quad (1)$$

According to Eq. 1, the inner eastern limit of the region is in $x = 1$. The model equation is:

$$\frac{\partial \psi}{\partial x} + \gamma\ J\left(\psi,\ \varsigma\right) - \kappa\gamma\ \nabla^2 \varsigma = 0. \tag{2}$$

In Eq. 2:

$$\varsigma = \nabla^2 \psi,\ \ \gamma = \beta^{-1},\ \ \kappa = \mathrm{Re}^{-1},\ \ \nabla^2 \varsigma = \alpha^2 \frac{\partial^2 \varsigma}{\partial x^2} + \frac{\partial^2 \varsigma}{\partial y^2}. \tag{3}$$

In turn, β is the dimensionless planetary vorticity gradient, Re is the Reynolds number and α is the ratio of north-south basin extent to east-west extent. In Eq. 3:

$$\beta = \beta_0 L^2 / U,\ \ \mathrm{Re} = UL / A_h \tag{3a}$$

where β_0 is the dimensional planetary vorticity gradient, whose order of magnitude is $10^{-11} m^{-1} s^{-1}$, and $A_h \leq 10^4 m^2 s^{-1}$ is the horizontal eddy viscosity (see e.g. Pedlosky, 1987). Given the conventional scale of the horizontal length, say $L = 10^6 m$, and of the current, say $U = 10^{-2} ms^{-1}$, from Eq. 3a we have $O(\beta) = 10^3$, $O(\mathrm{Re}) \geq 1$. Hence:

$$O\left(\gamma\right) = 10^{-3},\ \ O\left(\kappa\gamma\right) \leq 10^{-3}. \tag{4}$$

Moreover $\alpha = L_y / L_x$ is the ratio of N-S basin extent to E-W extent. In Ierley's model, α runs from 3.0 to 20 but a specific value of α is inessential in our analytical context. The boundary conditions are those of no mass-flux:

$$\psi\left(-1, y\right) = \psi\left(x, \pm 1\right) = 0 \tag{5}$$

supplemented by the additional (free-slip) conditions:

$$\varsigma\left(-1, y\right) = \varsigma\left(x, \pm 1\right) = 0. \tag{6}$$

At the open boundary $x = 1$ the matching between the solution $\psi(x, y)$ inside Q and the Sverdrup solution $\psi_s(x, y)$ outside Q takes place, that is to say:

$$\psi\left(1, y\right) = \psi_{Sv}\left(1, y\right),\ \ \varsigma\left(1, y\right) = \varsigma_{Sv}\left(1, y\right). \tag{7}$$

The Sverdrup solution is singled out in terms of the wind-stress curl, say $F(y)$, and the longitude of the eastern boundary, say $x_0 > 1$, where $\psi_{Sv}(x_0, y) = 0$. In the case of a single gyre, $F(y)$ must satisfy the conditions:

$$O(F) = 1,\ F(-1) = F(1) = 0 \tag{8}$$

and

$$F(y) > 0\ \ \forall y \in \left]-1,1\right[\tag{9a}$$

or

$$F(y) < 0\ \ \forall y \in \left]-1,1\right[. \tag{9b}$$

In particular, if according to Eq. 9b:

$$\begin{aligned} F(y) &= -\cos(\pi y/2) \\ x_0 &= 3 \end{aligned} \tag{10}$$

the integration of the Sverdrup balance $\dfrac{\partial \psi_{Sv}}{\partial x} = F(y)$ gives:

$$\psi_{Sv}(x,y) = (3-x)\cos(\pi y/2) \tag{11}$$

whence

$$\varsigma_{Sv}(x,y) = \nabla^2 \psi_{Sv} = (x-3)(\pi/2)^2 \cos(\pi y/2). \tag{12}$$

By using Eqs. 11 and 12, the matching conditions are just those of Ierley, that is:

$$\psi(1,y) = 2\cos(\pi y/2) \tag{13}$$

and

$$\varsigma(1,y) = -(\pi^2/2)\cos(\pi y/2). \tag{14}$$

Once problem identified by Eqs. 2, 5, 6, 13, and 14 has been established, it can be solved numerically, as Ierley did. A distinctive feature of this kind of model is that neither nonlinearity nor dissipation affect the Sverdrup-like behaviour of the solution outside Q. Note that, because of Eq. 11, $\psi_{Sv}(x, y) = \psi_{Sv}(x, -y)$ while nonlinearity breaks the symmetry $\psi(x, y) = \psi(x, -y)$ for $(x, y)\in Q$. However the matching condition in Eq. 7 demands $\psi(1, y) = \psi(1, -y)$ just like ψ_{Sv}. Presumably the latter constraint is satisfied only by special values of γ, κ and α, as anticipated.

3. Ieley's regional model implies that the intensification of the wind-driven current is necessarily westward

Here we reconsider Ierley's model but with a generic forcing $F(y)$ satisfying Eqs. 8 and 9, in place of that sinusoidal adopted by that author. Thus:

$$\psi_{Sv} = \left(x-3\right)F\left(y\right) \tag{15}$$

in place of Eq. 11 and, hence:

$$\varsigma_{Sv} = \left(x-3\right)F_{yy} \tag{16}$$

in place of Eq. 12. Given the boundary conditions in Eqs. 5 and 6, the matching conditions in Eq. 7, and the Sverdrupian fields in Eqs. 15 and 16, the energetics associated to Eq. 2 is evaluated by multiplying Eq. 2 by ψ and integrating the products on Q. The resulting equation, derived with full details in Appendix A, turns out to be:

$$\int_{-1}^{1}\left\{2F^2+4\gamma\left(F_y\right)^3\right\}dy-\kappa\gamma\int_Q \varsigma^2\,dx\,dy=0. \tag{17}$$

Eq. 17 is mathematically self consistent. Physically we highlight that, because of the first inequality of Eq. 4, the order of magnitude of the term $\int_{-1}^{1}\left\{2F^2+4\gamma\left(F_y\right)^3\right\}dy$ is the same as $2\int_{-1}^{1}F^2dy$, i.e. one. If, for mathematical simplicity, $F(y)$ is assumed to be an even function as Ierley did, then $\int_{-1}^{1}\left(F_y\right)^3dy=0$ and Eq. 17 simplifies as a consequence. In any case, also the last term of Eq. 17 must be of order one, in spite of the second inequality of Eq. 4. This means that the order of magnitude of ς^2 must be large enough to compensate the smallness of the factor $\kappa\gamma$, as the second inequality of Eq. 4 claims. Condition $O(\varsigma^2)>1$ means, in the proximity of the rigid boundaries of Q, that:

$$O\left(\partial u/\partial y\right)>1,\ O\left(\partial v/\partial x\right)>1. \tag{18}$$

Relationships in Eq. 18 are nothing but conditions for the intensification of the current inside Q. This is in full accordance with Ierley's numerical results although the author does not dwell on this aspect. Pedlosky (1987) uses the same argument in dealing with dissipation integrals for steady circulation (section 5.9 of the cited textbook). In conclusion, Eq. 17 implies that the westward intensification does take place in the region Q where the governing equation is Eq. 2. In other words, the western collocation of the region Q, with respect to the full fluid domain, implies that Q hosts a westward intensified current field.

At this point we reject provisionally the collocation western of Q and investigate a model quite analogous to that of Ierley but in which all the constituents are mirror reflected around the longitude $x = 0$. Thus, Q retains its position but, now, it is located at the easternmost area of the basin while the Sverdrupian region extends from $x = –1$ down to $x = –3$ (see Fig. 2).

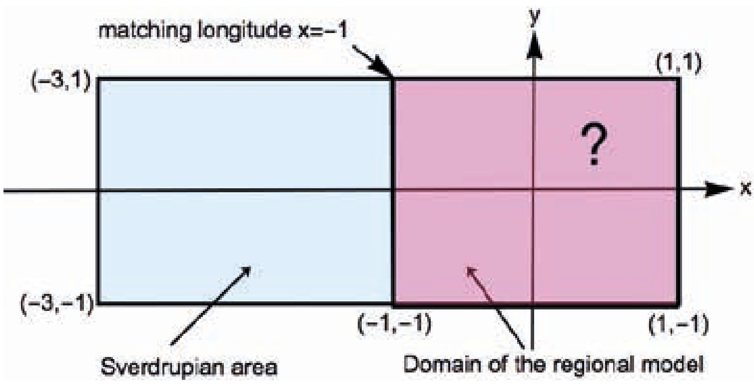

Fig. 2 - Sketch of the total fluid domain, with the specification of the region governed by Eq. 2 and the Sverdrupian area in which Eq. 19 holds. The question mark reminds the inconsistency of the model in the above shown configuration.

To avoid misunderstanding with the original model by Ierley, all the fields will be written with an overbar. In this case $\delta\bar{\Psi}_{Sv} / \delta x = F(y)$ and $\bar{\Psi}_{Sv}(–3, y) = 0$. Thus, in place of Eqs. 15 and 16, we have:

$$\bar{\psi}_{Sv} = \left(x+3\right)F\left(y\right) \tag{19}$$

and

$$\bar{\varsigma}_{Sv} = \left(x+3\right)F_{yy}. \tag{20}$$

Boundary conditions are:

$$\bar{\psi}\left(1,y\right) = \bar{\psi}\left(x,\ \pm 1\right) = 0,\ \bar{\varsigma}\left(1,y\right) = \bar{\varsigma}\left(x,\ \pm 1\right) = 0 \tag{21}$$

and the matching between $\bar{\Psi}$ and $\bar{\Psi}_{Sv}$ take the form:

$$\bar{\psi}(-1,y) = \bar{\psi}_{Sv}(-1,y), \ \bar{\varsigma}(-1,y) = \bar{\varsigma}_{Sv}(-1,y). \tag{22}$$

Given the boundary conditions in Eq. 21, the matching conditions of Eq. 22, and the Sverdrupian fields in Eqs. 19 and 20, the energetics associated to Eq. 2 and written in terms of $\bar{\psi}$, $\bar{\varsigma}$ is evaluated by multiplying Eq. 2 by $\bar{\psi}$ and integrating the products on Q with the aid of Eqs. 19, 20, 21, and 22. The resulting equation, derived in Appendix A, turns out to be:

$$\int_{-1}^{1} \left\{ -2F^2 + 4\gamma \left(F_y\right)^3 \right\} dy - \kappa\gamma \int_Q \bar{\varsigma}^2 \, dx\, dy = 0. \tag{23}$$

Eq. 23 differs from Eq. 17 only by the sign in front of F^2, but this change has a drastic consequence. In fact, if $F(y)$ were even, Eq. 23 would be false already from the mathematical point of view, since the sum of two negative quantities is not zero. On the other hand, if the contribution of $4\gamma \int_{-1}^{1} \left(F_y\right)^3 dy$ were different from zero, Eq. 23 could be mathematically consistent under the hypothesis $4\gamma \int_{-1}^{1} \left(F_y\right)^3 dy > 0$, but the first of inequalities in Eq. 4 prevents, on scaling arguments, the possibility of a balance of the kind:

$$4\gamma \int_{-1}^{1} \left(F_y\right)^3 dy = 2\int_{-1}^{1} F^2 \, dy + \kappa\gamma \int_Q \bar{\varsigma}^2 \, dx\, dy. \tag{24}$$

Even admitting, *ad absurdum*, an eastward intensification that would imply:

$$O\left(\kappa\gamma \int_Q \bar{\varsigma}^2 \, dx\, dy \right) = 1 \tag{25}$$

in any case the order of magnitude of the full right term of Eq. 24 is one (recall the first condition in Eq. 8). On the other hand, according to the first estimate in Eq. 4 and keeping in mind that (see Appendix B):

$$O\left(F_y\right) = 1 \tag{26}$$

we have:

$$O\left(4\gamma \int_Q \left(F_y\right)^3 dx\, dy \right) = 10^{-3}. \tag{27}$$

An evident imbalance between the left and the right terms of Eq. 24 turns out from Eqs. 25

and 27. This is due, ultimately, to the sign on front of F^2. Thus, the eastern collocation of the region Q, with respect to the full fluid domain, leads to a physical inconsistence and, therefore, intensification is necessarily westwards.

4. Conclusions

Ierley's model can be successfully adopted to prove that the intensification of wind-driven flows is necessarily westwards. Although this model, unlike those cites in the introduction, takes into account also the nonlinearity of the system through the Jacobian appearing in Eq. 2, our Eq. 17 shows that this aspect of the dynamics is secondary for two reasons:

- The standard forcing is usually an even functions of latitude, so $\int_{-1}^{1}\left(F_y\right)^3 dy = 0$ and, hence, the contribution of the non linear term appearing into I_1 (i.e. the second integral of Eq. A.1) is exactly zero;
- Even if $F(y)$ were not an even function, Eq. 17 shows that, because the first inequality in Eq. 4, $O\left(F^2\right) > O\left(2\gamma\left(F_y\right)^3\right)$. Thus, the main balance in Eq. 17 is in any case between the first and the third term, so nonlinearity is ruled out.

Of course, nonlinearity is essential to obtain the model solutions that produce, in particular, the recirculation in the north-western corner of the basin since the latter could not be obtained in a linear context. However the nonlinear term of Eq. 2, after integration on the sub-domain Q, gives a contribution to Eq. 17 that is inessential to decide between the validity of Eqs. 17 or 23. A glance to Eqs. A.21, A.16, and A.37 shows that only the first term of Eq. 2 is responsible of the crucial difference between Eqs. 17 and 23. This fact is perfectly in line with all the previous proofs of the westward nature of intensification, since each of them presupposes a non-vanishing planetary vorticity gradient, up from the archetypal model of Stommel (1948).

REFERENCES

Boening C.W.; 1986: *On the influence of frictional parameterization in wind-driven ocean circulation models*. Dyn. Atmos. Oceans, 18, 739-751, doi: 10.1016/0377-0265(86)90010-2.

Crisciani F. and Purini R.; 1997: *Boundary condition of Sverdrup solution from flow energetics*. J. Phys. Oceanogr., 27, 357-361, doi: 10.1175/1520-0485(1997)027<0357:BCOTSS>2.0.CO;2.

Crisciani F. and Mosetti R.; 2021: *Remarks on the westward intensification of wind-driven ocean transport*. Bull. Geoph. Ocean., 62, 577-594, doi: 10.4430/bgo00375.

Cushman_Roisin B.; 1994: *Introduction to geophysical fluid dynamics*. Prentice Hall, Englewood Cliffs, NJ, USA, 320 pp.

Ierley G.R.; 1987: *On the onset of inertial recirculation in barotropic general circulation models*. J. Phys. Oceanogr., 17, 2366-2374.

Pedlosky J.; 1965: *A note on western intensification of oceanic circulation*. J. Mar. Res., 23, 207-209.

Pedlosky J.; 1987: *Geophysical fluid dynamics*. Springer Verlag, Berlin, Germany, 710 pp.

Pedlosky J.; 1996: *Ocean circulation theory*. Springer Verlag, Berlin, Germany, 453 pp.

Pond S. and Pickard G.L.; 1983: *Introductory Dynamical Oceanography*. Pergamon Press, Oxford, UK, 329 pp.

Stommel H.; 1948: *The westward intensification of wind-driven ocean currents*. Trans. Am. Geophys. Union, 29, 202-206.

Welander P.; 1959: *On the vertically integrated mass transport in the ocean*. In: Bolib B. (ed), The atmosphere and sea in motion, Rockfeller Institute Press, New York, NY, USA, pp. 75-100.
Wunsh C.; 2011: *The decadal mean ocean circulation and Sverdrup balance*. J. Mar. Res., 69, 417-434.

Corresponding author: Renzo Mosetti
E-mail: rmosetti@gmail.com

Appendix A: Computation of the final Eqs. 17 and 23 from the energetics of the regional area

We premise that, unlike the numerical approach carried out by Ierley, the analytical development of the integrated energy balance (see below) needs also the continuity of the partial derivatives of the fields in $x = 1$:

$$\left(\frac{\partial \psi}{\partial x}\right)_{x=1} = \left(\frac{\partial \psi_{Sv}}{\partial x}\right)_{x=1} = F(y),\ \left(\frac{\partial \psi}{\partial y}\right)_{x=1} = \left(\frac{\partial \psi_{Sv}}{\partial y}\right)_{x=1} = -2F_y$$
$$\left(\frac{\partial \varsigma}{\partial x}\right)_{x=1} = \left(\frac{\partial \varsigma_{Sv}}{\partial x}\right)_{x=1} = F_{yy} \tag{A.1}$$

in addition to the boundary and matching conditions listed in section 2. The starting equation is the energy balance of the regional model:

$$\underbrace{\int_Q \frac{\partial}{\partial x}\psi^2\,dx\,dy}_{I_1} + \underbrace{\gamma\int_Q J\left(\psi^2,\varsigma\right)dx\,dy}_{I_2} - \underbrace{2\kappa\gamma\int_Q \nabla^2\varsigma\,dx\,dy}_{I_3} = 0. \tag{A.2}$$

⇒ Consider first the case in which Q is in the extreme western area of the basin, so the Sverdrupian part of the ocean is on its right. Then, because Eqs. 5, 7, and 15 one obtains:

$$I_1 = \int_{-1}^{1}\psi^2(1,y)\,dy = \int_{-1}^{1}\psi_{Sv}^2(1,y)\,dy = 4\int_{-1}^{1}F^2(y)\,dy. \tag{A.3}$$

To compute I_2 the identity $J(a,b) = \frac{\partial}{\partial y}\left(b\frac{\partial a}{\partial x}\right) - \frac{\partial}{\partial x}\left(b\frac{\partial a}{\partial y}\right)$ is in order. Hence:

$$I_2 = \gamma \int_{-1}^{1} \left[\varsigma \frac{\partial}{\partial x} \psi^2 \right]_{y=-1}^{y=1} dx - \gamma \int_{-1}^{1} \left[\varsigma \frac{\partial}{\partial y} \psi^2 \right]_{x=-1}^{x=1} dy. \tag{A.4}$$

Owing to Eqs. 5 and 6, Eq. A.4 simplifies into:

$$I_2 = -\gamma \int_{-1}^{1} \varsigma(1,y) \frac{\partial}{\partial y} \psi^2(1,y) dy. \tag{A.5}$$

Substitution of the matching conditions in Eqs. 7 and A.1 into Eq. A.5 results in the equation:

$$I_2 = 16\gamma \int_{-1}^{1} F_{yy} F_y F \, dy. \tag{A.6}$$

Noting that $F_{yy}F = \frac{1}{2}\left(F_y^2\right)_y$ and using the boundary conditions in Eq. 8, an integration by parts of Eq. A.6 gives:

$$I_2 = 8\gamma \int_{-1}^{1} \left(F_y\right)^3 dy. \tag{A.7}$$

Integral I_3 can be written as:

$$I_3 = -2\kappa\gamma\alpha^2 \underbrace{\int_Q \psi \frac{\partial^2 \varsigma}{\partial x^2} \, dxdy}_{\Lambda_1} - 2\kappa\gamma \underbrace{\int_Q \psi \frac{\partial^2 \varsigma}{\partial y^2} \, dxdy}_{\Lambda_2}. \tag{A.8}$$

Consider separately Λ_1 and Λ_2: an integration by parts gives:

$$\Lambda_1 = \int_{-1}^{1} dy \left\{ \left[\psi \frac{\partial \varsigma}{\partial x} \right]_{x=-1}^{x=1} - \int_{-1}^{1} \frac{\partial \psi}{\partial x} \frac{\partial \varsigma}{\partial x} dx \right\}. \tag{A.9}$$

Because of Eq. 5, Eq. A.9 can be expanded as follows:

$$\Lambda_1 = \int_{-1}^{1} dy \;\; \psi(1,y) \left(\frac{\partial \varsigma}{\partial x} \right)_{x=1} \partial x - \int_Q \left\{ \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \varsigma \right) - \frac{\partial^2 \psi}{\partial x^2} \varsigma \right\} \, dxdy \tag{A.10}$$

where:

$$\int_{-1}^{1}\psi(1,y)\left(\frac{\partial\varsigma}{\partial x}\right)_{x=1}dy=\int_{-1}^{1}\psi_{Sv}(1,y)\left(\frac{\partial\varsigma_{Sv}}{\partial x}\right)_{x=1}dy=-2\int_{-1}^{1}F(y)F_{yy}dy \tag{A.11}$$

and

$$\begin{aligned}&-\int_Q\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\varsigma\right)dxdy=-\int_{-1}^{1}\left[\frac{\partial\psi}{\partial x}\varsigma\right]_{x=-1}^{x=1}dy=-\int_{-1}^{1}\left(\frac{\partial\psi_{Sv}}{\partial x}\right)_{x=1}\varsigma_{Sv}(1,y)dy\\&=2\int_{-1}^{1}F(y)\,F_{yy}dy\end{aligned} \tag{A.12}$$

Substitution of Eqs. A.11 and A.12 into Eq. A.10 results in the equation:

$$\Lambda_1=\int_Q\frac{\partial^2\psi}{\partial x^2}\,\varsigma\,dxdy. \tag{A.13}$$

Moreover, owing to Eq. 5:

$$\Lambda_2=\int_{-1}^{1}dx\left[\psi\frac{\partial\varsigma}{\partial y}\right]_{y=-1}^{y=1}-\int_Q\frac{\partial\psi}{\partial y}\frac{\partial\varsigma}{\partial y}dxdy=-\int_Q\frac{\partial\psi}{\partial y}\frac{\partial\varsigma}{\partial y}dxdy. \tag{A.14}$$

Integrating by parts the last term of Eq. A.14 yields:

$$\Lambda_2=-\int_{-1}^{1}dx\left[\frac{\partial\psi}{\partial y}\varsigma\right]_{y=-1}^{y=1}+\int_Q\frac{\partial^2\psi}{\partial y^2}\varsigma\,dxdy. \tag{A.15}$$

Because of Eq. 6, Eq. A.15 takes the final form:

$$\Lambda_2=\int_Q\frac{\partial^2\psi}{\partial y^2}\varsigma\,dxdy. \tag{A.16}$$

Putting together Eqs. A.8, A.13, and A.16, that is to say:

$$I_3=-2\kappa\gamma\alpha^2\int_Q\frac{\partial^2\psi}{\partial x^2}\,\varsigma\,dxdy-2\kappa\gamma\int_Q\frac{\partial^2\psi}{\partial y^2}\,\varsigma\,dxdy$$

$$= -2\kappa\gamma \int_Q \left(\alpha^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \varsigma \; dx\,dy$$

one obtains:

$$I_3 = -2\kappa\gamma \int_Q \varsigma^2 \, dx\,dy. \tag{A.17}$$

Finally, substitution of Eqs. A.3, A.7, and A.17 into Eq. A.2 gives:

$$\int_{-1}^{1} \left\{ 2F^2 + 4\gamma \left(F_y \right)^3 \right\} dy - \kappa\gamma \int_Q \varsigma^2 \, dx\,dy = 0 \tag{A.18}$$

that is Eq. 17.

⇒ Consider now the case in which Q is in the extreme eastern area of the basin, so the Sverdrupian part of the ocean is on its left, Eq. A.2 can be re-stated as:

$$\underbrace{\int_Q \frac{\partial}{\partial x} \bar{\psi}^2 \, dx\,dy}_{\bar{I}_1} + \underbrace{\gamma \int_Q J\left(\bar{\psi}^2, \bar{\varsigma} \right) dx\,dy}_{\bar{I}_2} - \underbrace{2\kappa\gamma \int_Q \nabla^2 \bar{\varsigma} \, dx\,dy}_{\bar{I}_3} = 0 \tag{A.19}$$

The overbars in Eq. A.19 mean that the fields are referred to the latter configuration rather than to the former, and therefore they obey to different matching and boundary conditions, as explained in the text. Moreover, in place of Eq. A.1 we have:

$$\left(\frac{\partial \bar{\psi}}{\partial x} \right)_{x=-1} = \left(\frac{\partial \bar{\psi}_{Sv}}{\partial x} \right)_{x=-1} = F(y), \; \left(\frac{\partial \bar{\psi}}{\partial y} \right)_{x=-1} = \left(\frac{\partial \bar{\psi}_{Sv}}{\partial y} \right)_{x=-1} = -2F_y$$
$$\left(\frac{\partial \bar{\varsigma}}{\partial x} \right)_{x=-1} = \left(\frac{\partial \bar{\varsigma}_{Sv}}{\partial x} \right)_{x=-1} = F_{yy} \tag{A.20}$$

where $\bar{\Psi}_{Sv} = (x + 3)\, F(y)$ and so on. Then, because Eqs. 5, 7, and 15, one obtains:

$$\bar{I}_1 = -\int_{-1}^{1} \psi^2 \left(-1, y \right) dy = -\int_{-1}^{1} \psi_{Sv}^2 \left(-1, y \right) dy = -4 \int_{-1}^{1} F^2 \left(y \right) dy. \tag{A.21}$$

Comparison of Eq. A.21 with Eq. A.3 shows that:

$$\bar{I}_1 = -I_1 \tag{A.22}$$

In analogy with Eq. A.4:

$$\overline{I}_2 = \gamma \int_{-1}^{1} \left[\overline{\varsigma} \frac{\partial}{\partial x} \overline{\psi}^2 \right]_{y=-1}^{y=1} dx - \gamma \int_{-1}^{1} \left[\overline{\varsigma} \frac{\partial}{\partial y} \overline{\psi}^2 \right]_{x=-1}^{x=1} dy. \tag{A.23}$$

Owing to Eq. 21, Eq. A.23 simplifies into:

$$\overline{I}_2 = \gamma \int_{-1}^{1} \overline{\varsigma}(-1, y) \frac{\partial}{\partial y} \overline{\psi}^2 (1, y) dy. \tag{A.24}$$

Substitution of the matching conditions in Eqs. 22 and A.20 into Eq. A.24 results in the equation:

$$\overline{I}_2 = 16\gamma \int_{-1}^{1} F_{yy} F_y F \, dy \tag{A.25}$$

that is

$$\overline{I}_2 = 8\gamma \int_{-1}^{1} \left(F_y\right)^3 dy. \tag{A.26}$$

Comparison of Eq. A.26 with Eq. A.7 shows that:

$$\overline{I}_2 = I_2. \tag{A.27}$$

Integral I_3 can be written as:

$$\overline{I}_3 = -2\kappa\gamma\alpha^2 \underbrace{\int_Q \overline{\psi} \frac{\partial^2 \overline{\varsigma}}{\partial x^2} \, dxdy}_{P_1} - 2\kappa\gamma \underbrace{\int_Q \overline{\psi} \frac{\partial^2 \overline{\varsigma}}{\partial y^2} \, dxdy}_{P_2}. \tag{A.28}$$

Consider separately P_1 and P_2: an integration by parts gives:

$$P_1 = \int_{-1}^{1} dy \left\{ \left[\overline{\psi} \frac{\partial \overline{\varsigma}}{\partial x} \right]_{x=-1}^{x=1} - \int_{-1}^{1} \frac{\partial \overline{\psi}}{\partial x} \frac{\partial \overline{\varsigma}}{\partial x} dx \right\}. \tag{A.29}$$

Because of Eq. 21, Eq. A.29 can be expanded as follows:

$$P_1 = -\int_{-1}^{1} dy\ \bar{\psi}(-1,y)\left(\frac{\partial \bar{\varsigma}}{\partial x}\right)_{x=-1} - \int_Q \left\{\frac{\partial}{\partial x}\left(\frac{\partial \bar{\psi}}{\partial x}\bar{\varsigma}\right) - \frac{\partial^2 \bar{\psi}}{\partial x^2}\bar{\varsigma}\right\} dxdy \quad \text{(A.30)}$$

where, according to Eqs. 15, 21, and A.20:

$$-\int_{-1}^{1} \bar{\psi}(-1,y)\left(\frac{\partial \bar{\varsigma}}{\partial x}\right)_{x=-1} dy = -\int_{-1}^{1} \bar{\psi}_{Sv}(-1,y)\left(\frac{\partial \bar{\varsigma}_{Sv}}{\partial x}\right)_{x=-1} dy = -2\int_{-1}^{1} F(y) F_{yy} dy \quad \text{(A.31)}$$

and

$$-\int_Q \frac{\partial}{\partial x}\left(\frac{\partial \bar{\psi}}{\partial x}\bar{\varsigma}\right) dxdy = -\int_{-1}^{1}\left[\frac{\partial \bar{\psi}}{\partial x}\bar{\varsigma}\right]_{=-1}^{x=1} dy = \int_{-1}^{1}\left(\frac{\partial \bar{\psi}_{Sv}}{\partial x}\right)_{x=-1} \bar{\varsigma}_{Sv}(-1,y) dy$$
$$= 2\int_{-1}^{1} F(y)\ F_{yy} dy \quad \text{(A.32)}$$

Substitution of Eqs. A.31 and A.32 into Eq. A.30 results in the equation:

$$P_1 = \int_Q \frac{\partial^2 \bar{\psi}}{\partial x^2}\ \bar{\varsigma}\ dxdy. \quad \text{(A.33)}$$

Moreover, owing to Eq. 21:

$$P_2 = \int_{-1}^{1} dx\left[\bar{\psi}\frac{\partial \bar{\varsigma}}{\partial y}\right]_{y=-1}^{y=1} - \int_Q \frac{\partial \bar{\psi}}{\partial y}\frac{\partial \bar{\varsigma}}{\partial y} dxdy = -\int_Q \frac{\partial \bar{\psi}}{\partial y}\frac{\partial \bar{\varsigma}}{\partial y} dxdy. \quad \text{(A.34)}$$

Integrating by parts the last term of Eq. A.34 yields:

$$P_2 = -\int_{-1}^{1} dx\left[\frac{\partial \bar{\psi}}{\partial y}\bar{\varsigma}\right]_{y=-1}^{y=1} + \int_Q \frac{\partial^2 \bar{\psi}}{\partial y^2}\bar{\varsigma}\ dxdy. \quad \text{(A.35)}$$

Because of Eq. 21, Eq. A.35 takes the final form:

$$P_2 = \int_Q \frac{\partial^2 \psi}{\partial y^2}\varsigma\ dxdy. \quad \text{(A.36)}$$

Putting together Eqs. A.28, A.33, and A.36, that is to say:

$$\bar{I}_3 = -2\kappa\gamma\alpha^2 \int_Q \frac{\partial^2 \bar{\psi}}{\partial x^2}\, \bar{\varsigma}\; dxdy - 2\kappa\gamma \int_Q \frac{\partial^2 \bar{\psi}}{\partial y^2}\, \bar{\varsigma}\; dxdy$$
$$= -2\kappa\gamma \int_Q \left(\alpha^2 \frac{\partial^2 \bar{\psi}}{\partial x^2} + \frac{\partial^2 \bar{\psi}}{\partial y^2} \right) \bar{\varsigma}\; dx\,dy$$

one obtains:

$$\bar{I}_3 = -2\kappa\gamma \int_Q \bar{\varsigma}^2\, dx\,dy. \tag{A.37}$$

Comparison of Eq. A.37 with Eq. A.17 shows that:

$$\bar{I}_3 = I_3. \tag{A.38}$$

Finally, substitution of Eqs. A.21, A.26, and A.37 into Eq. A.19 gives:

$$\int_{-1}^{1} \left\{ -2F^2 + 4\gamma \left(F_y \right)^3 \right\} dy - \kappa\gamma \int_Q \bar{\varsigma}^2\, dx\,dy = 0 \tag{A.39}$$

that is Eq. 23.

Appendix B: Proof of Eq. 26

Eq. 19 yields the following dimensionless components of the Sverdrup current ($\bar{u}_{Sv}$, $\bar{v}_{Sv}$):

$$\bar{u}_{Sv} = -\left(x+3\right) F_y,\; \bar{v}_{Sv} = F\left(y\right). \tag{B.1}$$

In terms of the longitudinal averaging $\left\langle \cdot \right\rangle = \frac{1}{2} \int_{-3}^{-1} \left(\cdot \right)$ one finds from Eq. B.1:

$$\left\langle \bar{u}_{Sv} \right\rangle = -F_y,\; \left\langle \bar{v}_{Sv} \right\rangle = F\left(y\right). \tag{B.2}$$

The isotropic nature of the motion in the Sverdrupian area, where the flow is not intensified, implies $O\left(\left\langle \bar{u}_{Sv} \right\rangle\right) = O\left(\left\langle \bar{v}_{Sv} \right\rangle\right)$, whence Eq. B.2 gives:

$$O\left(F_y\right) = O\left(F\right). \tag{B.3}$$

In turn, because of the first condition of Eq. 8, Eq. B.3 implies:

$$O\left(F_y\right)=1. \tag{B.4}$$

Eq. B.4 prevents that the integrand at the left side of Eq. 24 may take values high enough to compensate the smallness of the factor 4*γ*, so as to equal its right and left sides.