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Remarks on the westward intensification of wind-driven ocean transport

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ABSTRACT A new reading of westward intensification is put forward by decomposing the transport stream function in the sum of a couple of symmetric and anti-symmetric terms, relatively to the mirror reflection with respect to the central longitude of a reference basin, rectangular in shape and included into a certain beta plane. The consequence of the decomposition is twofold: first, the anti-symmetric term turns out to be a strict consequence of the latitudinal dependence of Coriolis acceleration, in the sense that it disappears in a hypothetical f-plane; second, whatever the wind forcing may be, the symmetric and anti-symmetric components of the meridional transport are always parallel in the western region of the basin and anti parallel in the eastern one. As a consequence of their superposition, the meridional transport is more energetic into the western region and it gives rise to the phenomenon of westward intensification. On the other hand, the components of the meridional transport tend to compensate each other in the eastern region in accordance with the Sverdrup balance.

Key words: wind-driven circulation, westward intensification, symmetry properties.

1. Introduction

The westward intensification of the wind-driven oceanic mass transport was first observed by the Portuguese explorer Ponce de Leon on 1513, in sailing countercurrent the Gulf Stream with unexpected difficulty, during the discovery of Florida. This phenomenon was explained only several centuries after in a seminal model of the American oceanographer Henry Stommel (1948), by resorting, as key ingredient, to the latitude dependence of Coriolis acceleration. His model was linear, referred to an idealised subtropical gyre subject to a sinusoidal wind-stress and with an *ad hoc* parameterisation of friction. A more realistic linear model was formulated later on by Munk (1950) who was able to refer it, with a certain success, to a number of real, and westward intensified, oceanic current fields. A qualitative explanation of the westward intensification, probably the first one, was put forward by Stommel (1958) for a steady ocean subject to a vorticity input by a large gyre-like wind field over it and an unspecified frictional mechanism. Linear models lead to a schematic partition of the gyre into a small western region with an intense transport and a large Sverdrup flow in the interior. A part from the details of the western area, in particular of the northern-western part, they revealed themselves to be consistent with observations, as confirmed, for instance, by Welander (1959) and Wunsch (2011). The more recent numerical investigations of nonlinear models, with the possibility to consider different additional boundary conditions, focused the attention towards the northward and eastward migration of the intensified current because of the inertial effect of non-linearity and their dynamic stability; see Pedlosky (1996) for references and a review. Indeed, the number of papers devoted primarily to the cause of westward intensification, detached from any particular model, is relatively small and researchers seem to have taken away further interest on the topic, so some gaps in the knowledge might still be there. On this subject, the paper of Pedlosky (1965) is particularly worth of mentioning for its elegance and generality. On the other hand, Crisciani and Purini (1997) used energy arguments and boundary layer techniques to prove the vanishing of Sverdrup solution at the eastern boundary of every basin and, hence, the westward localisation of intensification. Arguments involving the dynamics of the bottom Ekman layer (Cushman-Roisin, 1994) demand too stringent assumptions about the meridional transport and are less convincing. Given that wind driven oceanic circulation is cast into gyres determined by the wind stress curl over the ocean (with the exception of the circumpolar current), the quasi-geostrophic dynamics of each gyre can be described with the aid of the beta plane approximation, of the traditional approximation of Coriolis acceleration and with the assumption of a latitudinal modulation of the wind-stress. In turn, the ordering parameter, by means of which the quasi-geostrophic governing equation is derived, is far larger (about times) than the Rossby number and this fact linearises the dynamics. Even if so simplified, gyre models always exhibit two fundamental dynamic balances: one between the Coriolis acceleration and the lateral diffusion of relative vorticity, the other between the Coriolis acceleration and the wind forcing. This is clearly schematised in Eq. 5.5.29 of Pedlosky (1987). The consequence of the first balance is the formation of the westward intensification of the mass transport in the western region of any gyre, while the second one is the fundamental Sverdrup balance that holds throughout the remaining central and eastern portion of the gyre. As a matter of fact, the intensification of the transport is somehow trapped in the proximity of the western boundary even if neither the Coriolis acceleration nor the wind-stress curl, depend on longitude. Indeed, the first balance demands that a ‘small’ longitudinal length scale and a related velocity be locally established, in the proximity of the western boundary, in order that dissipation growths enough to be comparable with Coriolis acceleration. Hence, the asymmetry of the intensification is explained in terms of the double longitudinal length scales along the basin.

2. Governing equations

With reference to a certain beta plane, the oceanic water body, extended below the surface Ekman layer down to the sea floor, is governed by the steady quasi-geostrophic vorticity equation:

$$\beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho} \hat{\mathbf{k}} \cdot rot \ \boldsymbol{\tau} + A_h \nabla^4 \psi . \tag{1}$$

In Eq. 1, β is the planetary vorticity gradient, ψ is the transport stream function, ρ is a constant reference density, $\boldsymbol{\tau}$ is the wind stress acting at the ocean surface and $A_h > 0$ is a turbulent viscosity coefficient. Eq. 1 is found, for instance, in Vallis (2006), his Eq. 14.42, while its foundation is discussed with full details by Pedlosky (1987), section 6.19. The transport stream function is related to the transport vector $\mathbf{M}$ by the relationship:

$$\nabla\psi = \mathbf{M}\times\hat{\mathbf{k}} \tag{2}$$

where:

$$\mathbf{M} = \left(M_x,\ M_y\right) = \left(\int_{-H}^{0} u\,dz, \int_{-H}^{0} v\,dz\right). \tag{3}$$

Eqs. 2 and 3 openly imply that the approach based on Eq. 1 is fully barotropic.

The scalar components of Eq. 2 are $\frac{\partial\psi}{\partial x} = M_y$, $\frac{\partial\psi}{\partial y} = -M_x$. In Eq. 3 $\left(u,v\right)$ is the geostrophic current and H is the thickness of the fluid layer governed by Eq. 1. Strictly speaking, the surface placed in $z=0$ separating the bottom of the surface Ekman layer from the interior is weakly spatially modulated, however this fact is irrelevant at the first order dynamics (Eq. 1) as shown, for instance, in Cavallini and Crisciani (2013). The following orders of magnitude, in SI units, taken from Pedlosky (1987), are fit for the system under investigation:

$$L=10^6,\ U=2\times10^{-3},\ H=5\times10^3,\ \beta=10^{-11},\ \tau=10^{-1},\ A_h=10\div10^4. \tag{4}$$

In Eq. 4, is the horizontal length scale and is the horizontal current. Evaluation of the parameter $\beta' = \beta L^2/U$ by means of Eq. 4 yields $\beta' = 5\times10^3$, that is to say $O\left(\beta'\right) >> 1$, and hence the net dominance of the Coriolis acceleration holds with respect to the local one, consistently with the linearity of the differential Eq. 1. By using Eq. 4 to evaluate the magnitudes of the terms of Eq. 1, one can ascertain that the last of them is several orders lesser than the other two which, in turn, have the same order and give rise to the Sverdrup balance:

$$\beta\frac{\partial\bar{\psi}}{\partial x} = \frac{1}{\rho}\hat{\mathbf{k}}\cdot rot\ \boldsymbol{\tau}. \tag{5}$$

Here, and in what follows, the overbar means that the related quantity refers to the Sverdrup balance. It is well known that Eq. 5 is accurate for the most of the wind-driven ocean [see, for instance, Welander (1959) and Wunsch (2001)] but it is unable to produce closed transport streamlines, as steady circulation in a bounded basin demands. Thus, the third term of Eq. 1 must be involved in the first order dynamics of the gyres, but with local scales L and U different from Eq. 4.

Setting, in short,

$$F\left(y\right) = \hat{\mathbf{k}}\cdot rot\ \boldsymbol{\tau} \tag{6}$$

the latitudinal shape of the wind field curl exhibits a sequence of zeros at definite circles of latitude and, hence, a constant sign between a couple of consecutive zeros. Let $y=0$ and $y=\ell$ be such a couple. Then the alternative is:

$$\begin{matrix} F(0)=F(\ell)=0 \\ F(y)>0 \ \ \forall y\in]0,\ell[\end{matrix} \quad \text{or} \quad \begin{matrix} F(0)=F(\ell)=0 \\ F(y)<0 \ \ \forall y\in]0,\ell[\end{matrix}. \tag{7}$$

In both cases (Eq. 7) the gyre is "packaged" into the interval $[0,\ell]$. The longitudinal extension of the gyre is ideally determined by two material boundaries, say at $x=-\lambda$ and $x=\lambda$ for mathematical convenience, representing the solid profiles of continental coastlines. Thus, the fluid domain Q is given by:

$$Q=\{(x,y):\ |x|\le\lambda,\ 0\le y\le\ell\}. \tag{8}$$

The assumed impermeability of the boundary ∂Q at every depth implies, by definition,

$$\mathbf{M}\cdot\hat{\mathbf{n}}=0 \ \ \forall(x,y)\in\partial Q \tag{9}$$

where $\hat{\mathbf{n}}$ is the unit vector normal, point by point, to ∂Q. Because Q is simply connected, condition in Eq. 9 is equivalent to:

$$\psi=0 \ \ \forall(x,y)\in\partial Q. \tag{10}$$

Besides Eq. 10, additional boundary conditions are requested to single out a unique solution of Eq. 1. In $x=-\lambda$ and in $x=\lambda$ the frictional retardation caused by the interaction of the flow with the solid boundaries testifies in favor of the standard condition of no slip, i.e.:

$$M_y(-\lambda,y)=0,\ M_y(\lambda,y)=0$$

whence

$$\frac{\partial}{\partial x}\psi(-\lambda,y)=0,\ \frac{\partial}{\partial x}\psi(\lambda,y)=0 \quad \forall y\in[0,\ell]. \tag{11}$$

In $y=0$ and in $y=\ell$, owing to the atmospheric origin of the zonal boundaries, the situation is less definite and two possibilities are considered, i.e. the condition of no slip, analogous to Eq. 11,

$$\frac{\partial}{\partial y}\psi\left(x,\ell\right)=0,\ \frac{\partial}{\partial y}\psi\left(x,0\right)=0\quad \forall\left|x\right|\le\lambda \tag{12}$$

or, alternatively, that of free slip, i.e.

$$\frac{\partial}{\partial x}M_y\left(x,\ell\right)-\frac{\partial}{\partial y}M_x\left(x,\ell\right)=0,\ \frac{\partial}{\partial x}M_y\left(x,0\right)-\frac{\partial}{\partial y}M_x\left(x,0\right)=0$$

whence

$$\nabla^2\psi\left(x,\ell\right)=0,\ \nabla^2\psi\left(x,0\right)=0\quad \forall\left|x\right|\le\lambda\ . \tag{13}$$

Note that Eq. 13 implies

$$\nabla^2\frac{\partial}{\partial x}\psi\left(x,\ell\right)=0,\ \nabla^2\frac{\partial}{\partial x}\psi\left(x,0\right)=0\ \ \forall\left|x\right|\le\lambda\ . \tag{14}$$

Third order relationships in Eq. 14 are suitable additional boundary conditions for the fourth order Eq. 1 as well.

3. Decomposition of the fields in symmetric and anti symmetric terms

Consider the transformation of Q into itself defined as the mirror reflection of each point with respect to the central longitude, that is:

$$\left(x,y\right)\mapsto\left(-x,y\right). \tag{15}$$

By means of Eq. 15 the scalar fields ψ, M_x, M_y, the boundary conditions in Eqs. 10 to 14 and differential operators can be decomposed in the sum of a symmetric and an anti-symmetric part, according to the procedure below described. Starting from the auxiliary function $\overleftarrow{\psi}\left(x,y\right)=\psi\left(-x,y\right)$, the fields

$$\psi^s\left(x,y\right)=\frac{1}{2}\left[\psi\left(x,y\right)+\overleftarrow{\psi}\left(x,y\right)\right],\ \psi^a\left(x,y\right)=\frac{1}{2}\left[\psi\left(x,y\right)-\overleftarrow{\psi}\left(x,y\right)\right] \tag{16}$$

are introduced. By applying Eq. 15 to Eq. 16 one finds:

$$\psi^s\left(-x,y\right)=\psi^s\left(x,y\right),\ \psi^a\left(-x,y\right)=-\psi^a\left(x,y\right) \tag{17}$$

so ψ^s and ψ^a are the symmetric and anti-symmetric constituents of ψ

$$\psi = \psi^s + \psi^a \,. \tag{18}$$

Quite analogously:

$$M_x = M_x^s + M_x^a,\;\; M_y = M_y^s + M_y^a \,. \tag{19}$$

Note also the useful identity

$$\psi^a\left(0, y\right) = 0 \,. \tag{20}$$

With little algebra one can ascertain that boundary conditions of the same form as Eqs. 10 to 14 hold true separately for ψ^s and for ψ^a. Transformation in Eq. 15 implies:

$$\frac{\partial}{\partial x} \mapsto -\frac{\partial}{\partial x},\;\; \frac{\partial}{\partial y} \mapsto \frac{\partial}{\partial y} \tag{21}$$

and hence the invariance of ∇^{2n} (n= 1, 2, ...) under Eq. 21 follows. Finally, from Eqs. 2, 18, 19, and 21, the equations:

$$\frac{\partial \psi^s}{\partial x} = M_y^a,\;\; \frac{\partial \psi^s}{\partial y} = -M_x^s,\;\; \frac{\partial \psi^a}{\partial x} = M_y^s,\;\; \frac{\partial \psi^a}{\partial y} = -M_x^a \tag{22}$$

can be inferred.

4. Results and discussion

The present investigation highlights a point of view that, in our opinion, is different of each the previous ones. Under the usual assumption of fluid domain (Eq. 8) rectangular in shape, the basic idea is to express the transport stream function as the sum of two terms, one symmetric under the mirror reflection with respect to the central longitude of the basin, the other anti-symmetric, according to Eq. 18. After little algebra an equation will be derived, independent of the forcing, which shows that the anti-symmetric constituent is non-vanishing if and only if the planetary vorticity gradient enters explicitly into the dynamics at the leading order. If planetary vorticity is a constant, the transport stream function is symmetric in the sense said above. In this perspective, the influence of a latitude-dependent Coriolis acceleration extends throughout the longitudinal domain of the gyre. The superposition of the constituents is constructive in the western domain and destructive in the eastern one. The consequent amplification in the western region produces the westward intensification, while in the eastern region the terms balance each other, in accordance with Sverdrup's relationship. Moreover, from equation above an inequality involving the product of the symmetric and anti symmetric components of the meridional transport will be inferred. It states that these components are necessarily parallel in the western region of the domain and anti parallel in the eastern one. This result implies that the meridional transport is more energetic in the western region than in that eastern, independently of the form of the forcing. In other words, intensification is always westward. Here, symmetry arguments are used to explain the basic features of westward intensification. After these preliminaries, note that, with notation in Eq. 6, Eq. 1 takes the form:

$$\beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho} F\left(y\right) + A_h \nabla^4 \psi \,. \tag{23}$$

By using Eq. 18, Eq. 23 becomes:

$$\beta \frac{\partial \psi^s}{\partial x} + \beta \frac{\partial \psi^a}{\partial x} = \frac{1}{\rho} F\left(y\right) + A_h \nabla^4 \psi^s + A_h \nabla^4 \psi^a \,. \tag{24}$$

Application of Eqs. 15 to 24 with the aid of Eqs. 17 and 21 gives:

$$-\beta \frac{\partial \psi^s}{\partial x} + \beta \frac{\partial \psi^a}{\partial x} = \frac{1}{\rho} F\left(y\right) + A_h \nabla^4 \psi^s - A_h \nabla^4 \psi^a \,. \tag{25}$$

Then, subtraction of Eq. 25 from Eq. 24 results in the equation:

$$\beta \frac{\partial \psi^s}{\partial x} = A_h \nabla^4 \psi^a \tag{26}$$

or, by using the first part of Eq. 22,

$$\beta \; M_y^a = A_h \nabla^4 \psi^a \,. \tag{27}$$

Multiplication of each term of Eq. 27 by $\frac{\partial \psi^a}{\partial x} = M_y^s$ yields:

$$\beta \; M_y^s M_y^a = A_h \frac{\partial \psi^a}{\partial x} \nabla^4 \psi^a \,. \tag{28}$$

Eqs. 26 and 28 are the starting points to derive the main results anticipated in the Introduction.

4.1. A consequence of Eq. 26

The aim of this subsection is to prove that the planetary vorticity gradient involves presence of an anti-symmetric term of the transport stream function

$$\beta \neq 0 \Rightarrow \psi^a \neq 0 \tag{29}$$

and, inversely to Eq. 29, that the anti-symmetric term presupposes a non-vanishing planetary vorticity gradient

$$\psi^a \neq 0 \Rightarrow \beta \neq 0. \tag{30}$$

Implications in Eqs. 29 and 30 are logically equivalent to:

$$\psi^a = 0 \Leftrightarrow \beta = 0. \tag{31}$$

Before proving Eq. 31, a clarification is in order. The parameter $\beta = 2\Omega\cos(\phi_0)/R$ is intrinsically different of zero because of Earth's rotation ($\Omega > 0$) and the finiteness of Earth' radius ($R < \infty$) so hypothesis $\beta = 0$ looks meaningless. However, β enters actually into the vorticity equation provided that $O(\beta L^2/U) \geq 1$, that is to say for motions on horizontal length scale 'large enough' and horizontal velocity scales 'small enough'. Wherever these conditions are not satisfied, the smallness of $\beta L^2/U$ makes null the influence of β on the motion. This fact is formally equivalent to assume $\beta = 0$ since the beginning.

After this premise, consider again Eq. 26 and, in view of Eq. 31, take

$$\psi^a = 0 \tag{32}$$

Hypothesis in Eq. 32 implies $\beta = 0$ or $\frac{\partial \psi^s}{\partial x} = 0$. Because of Eq. 32, the latter is equivalent to $\frac{\partial \psi}{\partial x} = 0$, that is $M_y = 0$, so the transport should be strictly zonal throughout the basin. In particular it should be zero in $x = -\lambda$ and in $x = \lambda$ because of Eq. 9, so M_x turns out to be zero everywhere. Thus $\frac{\partial \psi^s}{\partial x} = 0$ implies $\mathbf{M} = \mathbf{0}$, but the inhomogeneity of Eq. 1 prevents an identically null transport. This proves that the possibility $\frac{\partial \psi^s}{\partial x} = 0$ must be rejected and therefore:

$$\psi^a = 0 \Rightarrow \beta = 0. \tag{33}$$

Take now

$$\beta = 0\ . \tag{34}$$

Hypothesis in Eq. 34 implies $A_h = 0$ or $\nabla^4 \psi^a = 0$. Consider first $A_h = 0$ and integrate Eq. 23 on Q:

$$\beta \int_Q \frac{\partial \psi}{\partial x} dx\,dy - \frac{1}{\rho} \int_Q F(y)\,dx\,dy = 0 \tag{35}$$

Green theorem together with Eq. 10 imply $\int_Q \frac{\partial \psi}{\partial x} dx\,dy = 0$ so Eq. 35 simplifies into: $\frac{1}{\rho} \int_Q F(y)\,dx\,dy = 0$, that is $\frac{2\lambda}{\rho} \int_0^{\ell} F(y)\,dy = 0$ but this equation is openly inconsistent with Eq. 7, and therefore $A_h \neq 0$. The remaining possibility is $\nabla^4 \psi^a = 0$. The goal is to prove that equation with boundary conditions:

$$\psi^a = 0 \ \ \forall (x, y) \in \partial Q \tag{36}$$

$$\frac{\partial}{\partial x} \psi^a(-\lambda, y) = 0,\ \frac{\partial}{\partial x} \psi^a(\lambda, y) = 0 \ \ \forall y \in [0, \ell] \tag{37}$$

and

$$\begin{aligned} &\frac{\partial}{\partial y} \psi^a(x,0) = 0,\ \frac{\partial}{\partial y} \psi^a(x,\ell) = 0 \\ &\mathit{or} \\ &\nabla^2 \psi^a(x,0) = 0,\ \nabla^2 \psi^a(x,\ell) = 0,\ \forall |x| \leq \lambda \end{aligned} \tag{38}$$

has the unique solution of Eq. 32. Note preliminarily that Eqs. 37 and 38 imply:

$$\nabla^2 \psi^a \ \ \nabla \psi^a \cdot \hat{\mathbf{n}} = 0 \ \ \forall (x, y) \in \partial Q\ . \tag{39}$$

Consider *ad absurdum* $\psi^a \neq 0$ and the identity

$$\psi^a \ \nabla^4\psi^a = \nabla\cdot\left[\psi^a \ \nabla\left(\nabla^2\psi^a\right)\right] - \nabla\psi^a\cdot\nabla\left(\nabla^2\psi^a\right) =$$
$$\nabla\cdot\left[\psi^a \ \nabla\left(\nabla^2\psi^a\right)\right] - \nabla\cdot\left(\nabla^2\psi^a \ \nabla\psi^a\right) + \left(\nabla^2\psi^a\right)^2 \tag{40}$$

The obvious implication $\nabla^4\psi^a = 0 \Rightarrow \int \psi^a \ \nabla^4\psi^a dxdy = 0$ and Eq. 40 give

$$\int_Q \left\{\nabla\cdot\left[\psi^a \ \nabla\left(\nabla^2\psi^a\right)\right] - \nabla\cdot\left(\nabla^2\psi^a \ \nabla\psi^a\right) + \left(\nabla^2\psi^a\right)^2\right\} dxdy = 0 \tag{41}$$

The first integral of Eq. 41 is zero because of Eq. 36, the second is zero owing to Eq. 39 and the third implies $\nabla^2\psi^a = 0$. In turn, the unique solution of problem

$$\begin{cases} \nabla^2\psi^a = 0 \ \ \forall\left(x,y\right)\in Q \\ \psi^a = 0 \ \ \forall\left(x,y\right)\in \partial Q \end{cases} \tag{42}$$

is Eq. 32. Thus $\beta = 0 \Rightarrow \psi^a = 0$ and Eq. 31 is fully proved.

4.2. A consequence of Eq. 28

Eq. 28 is anti symmetric and therefore:

$$\int_Q M_y^s M_y^a \ dxdy = 0\,. \tag{43}$$

Consider the partition of Q

Then, Eq. 43 implies

$$Q_1 = \left\{\left(x,y\right): \ \left(-\lambda \le x \le 0\right)\times\left(0 \le y \le \ell\right)\right\}, \ Q_2 = \left\{\left(x,y\right): \ \left(0 \le x \le \lambda\right)\times\left(0 \le y \le \ell\right)\right\}$$

$$\int_{Q_1} M_y^s M_y^a \ dxdy = -\int_{Q_2} M_y^s M_y^a \ dxdy\,. \tag{44}$$

In the remaining part of this subsection the inequality

$$\int_{Q_1} M_y^s M_y^a \ dxdy > 0 \tag{45}$$

will be proved. By using Eq. 45, the inequality

$$\int_{Q_1} M_y^2 \ dxdy - \int_{Q_2} M_y^2 \ dxdy = 4\int_{Q_1} M_y^s M_y^a \ dxdy > 0 \tag{46}$$

can be easily derived and, from Eq. 45 one finds:

$$\int_{Q_1} M_y^2 \ dxdy > \int_{Q_2} M_y^2 \ dxdy \ . \tag{47}$$

Inequality 47 shows that the meridional transport is more energetic in the western region of the basin than in the eastern one. Thus, Eq. 47 states that the intensified meridional transport is western, whatever the forcing (Eq. 6) may be. Moreover, Eq. 45 clarifies the mechanism of the intensification. In fact, the positive sign of Eq. 45 means that, in Q_1, the symmetric and the anti-symmetric components of the transport are predominantly parallel, so they sum each other, while the opposite happens in the eastern region where $\int_{Q_2} M_y^s M_y^a \ dxdy < 0$.

Recalling Eq. 28, the proof of Eq. 45 relies on the evaluation of the integral

$$I_1 = \int_0^{\ell} dy \int_{-\lambda}^{0} dx \frac{\partial \psi^a}{\partial x} \nabla^4 \psi^a \ . \tag{48}$$

Integral 48 can be carried out by means of the identity

$$\begin{aligned} \frac{\partial \psi^a}{\partial x} \nabla^4 \psi^a = \frac{\partial}{\partial x}\left(\psi^a \ \ \nabla^4 \psi^a\right) - \nabla \cdot \left[\psi^a \ \ \nabla\left(\nabla^2 \frac{\partial \psi^a}{\partial x}\right)\right] \\ + \nabla \cdot \left(\nabla \psi^a \ \ \nabla^2 \frac{\partial \psi^a}{\partial x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\nabla^2 \psi^a\right)^2. \end{aligned} \tag{49}$$

Owing to the anti symmetric part of boundary condition in Eq. 10 and to Eq. 20, the contributions of the first and of the second term at the r.h.s of Eqs. 48 and 49 are zero and Eq. 48 can be restated as:

$$I_1 = \int_0^{\ell} dy \left\{ \left[\frac{\partial \psi^a}{\partial x} \nabla^2 \frac{\partial \psi^a}{\partial x} \right]_{x=-\lambda}^{x=0} - \frac{1}{2} \left[\left(\nabla^2 \psi^a \right)^2 \right]_{x=-\lambda}^{x=0} \right\}$$
$$+ \int_{-\lambda}^{0} dx \left[\frac{\partial \psi^a}{\partial y} \nabla^2 \frac{\partial \psi^a}{\partial x} \right]_{y=0}^{y=\ell} . \tag{50}$$

Noting that $\frac{\partial}{\partial x} \psi^a(-\lambda, y) = 0$ because of Eq. 11, $\nabla^2 \psi^a(0, y) = 0$ according to Eq. 20 and $\frac{\partial}{\partial y} \psi^a(x, \ell) = 0$, $\frac{\partial}{\partial y} \psi^a(x, 0) = 0$ or $\nabla^2 \frac{\partial}{\partial x} \psi^a(x, \ell) = 0$, $\nabla^2 \frac{\partial}{\partial x} \psi^a(x, 0) = 0$ owing to Eqs. 12 and 14, respectively, Eq. 50 simplifies into:

$$I_1 = \int_0^{\ell} dy \left\{ \frac{\partial}{\partial x} \psi^a(0, y) \nabla^2 \frac{\partial}{\partial x} \psi^a(0, y) + \frac{1}{2} \left[\nabla^2 \psi^a(-\lambda, y) \right]^2 \right\} . \tag{51}$$

In terms of the transport, and by using the no mass flux boundary condition $\frac{\partial}{\partial y} M_x^a(-\lambda, y) = 0$, Eq. 51 takes the form:

$$I_1 = \int_0^{\ell} dy \left\{ M_y^s(0, y) \nabla^2 M_y^s(0, y) + \frac{1}{2} \left[\frac{\partial}{\partial x} M_y^s(-\lambda, y) \right]^2 \right\} . \tag{52}$$

The sign of the integrand of Eq. 52 can be determined on scaling arguments as follows. Because of mass conservation, $\int_{-\lambda}^{\lambda} M_y \, dx = 0$ and, in turn, this equation implies

$$\int_{-\lambda}^{0} M_y^s \, dx = 0 . \tag{53}$$

As anticipated in the comment to Eq. 5, in the most of the every gyre, say in aregion $R \subset Q$ of the kind

$$R = \left\{ (x, y) : \; (-\xi \le x \le \xi) \times (0 \le y \le \ell) \right\}$$

where $\left(-\xi,\ \xi\right) \subset \left(-\lambda,\ \lambda\right)$ and

$$O\left(\frac{\lambda-\xi}{\xi}\right) \leq 10^{-1}, \tag{54}$$

the Sverdrup balance (Eq. 5) is an accurate representation of the full Eq. 1. Note that the extension of $\left(-\xi,\ \xi\right)$ to $\left(-\xi,\ \lambda\right)$ cannot be considered here because it would be equivalent to state that $\bar{\psi}\left(\lambda,\ y\right)=0$ and, hence, to assume the westward intensification that, instead, must be proved. Thus, in $\left(-\xi,\ \xi\right)$

$$M_y^s \cong \bar{M}_y^s = \hat{\mathbf{k}} \cdot rot\ \boldsymbol{\tau} / \left(\beta\rho\right). \tag{55}$$

Keeping in mind Eq. 55, Eq. 53 can be written as:

$$\int_{-\lambda}^{-\xi} M_y^s dx + \int_{-\xi}^{0} \bar{M}_y^s dx = 0 . \tag{56}$$

Then, from Eqs. 54 and 56 the estimate

$$O\left(\bar{M}_y^s / \bar{M}_y^s\right) = O\left(\left(\lambda-\xi\right)/\xi\right) \tag{57}$$

is inferred. Going back to Eq. 52 and by identifying $M_y^s\left(0,y\right)=\bar{M}_y^s\left(0,y\right)$ in that $x=0 \in \left(-\xi,\ \xi\right)$, Eq. 52 becomes:

$$I_1 = \int_0^{\ell} dy \left\{ \bar{M}_y^s\left(0,y\right)\nabla^2 \bar{M}_y^s\left(0,y\right) + \frac{1}{2}\left[\frac{\partial}{\partial x} M_y^s\left(-\lambda,y\right)\right]^2 \right\}.$$

Noting that in I_1, $\nabla^2 = \frac{1}{4\xi^2}\nabla'^2$, and $\frac{\partial}{\partial x} = \frac{1}{\lambda-\xi}\frac{\partial}{\partial x'}$ where ∇' and x' are non-dimensional, the ratio of the orders of magnitude of the first integrand to the second is

$$2\frac{O\left(\bar{M}_y^s\ \ \nabla^2 \bar{M}_y^s\right)}{O\left(\left(\partial M_y^s / \partial x\right)^2\right)} = O\left(\left(\bar{M}_y^s / M_y^s\right)^2\right) O\left(\left(\frac{\lambda-\xi}{2\xi}\right)^2\right)$$

that is to say, recalling Eqs. 54 and 57, one derives the relationship:

$$2\frac{O\left(\bar{M}_y^s\ \nabla^2\bar{M}_y^s\right)}{O\left(\left(\partial M_y^s/\partial x\right)^2\right)}=O\left(\frac{1}{2}\left(\frac{\lambda-\xi}{\xi}\right)^4\right). \tag{58}$$

Because of Eq. 54, $\frac{1}{2}\left(\frac{\lambda-\xi}{\xi}\right)^4 \le 5\times 10^{-5}$ so Eq. 58 implies

$$I_1 \cong \frac{1}{2}\int_0^{\ell} dy\left[\frac{\partial}{\partial x}M_y^s\left(-\lambda,y\right)\right]^2>0 \tag{59}$$

and hence $\beta\int_{Q_1} M_y^s M_y^a\,dx\,dy \cong A_h I_1>0 \Rightarrow \int_{Q_1} M_y^s M_y^a\,dx\,dy>0$. Inequality 45 is so proved.

4.3. Application to a specific wind forcing

Two examples will be useful to explain inequality 45 and the subsequent inequality 47. A suitable non dimensional version of Eq. 1 is considered in the domain $Q=\left\{\left(x,y\right):\ \left|x\right|\le 1,\ 0\le y\le 1\right\}$, with forcing $F\left(y\right)=\pm\sin\left(\pi y\right)$. Fig. 1 refers to the case $F\left(y\right)=-\sin\left(\pi y\right)$, fit for an idealised subtropical gyre. Fig. 1a depicts an anti-cyclonic circulation system determined by the symmetric stream function $\psi^s\left(x,y\right)$, while Fig. 1b shows a couple cyclonic, anti-cyclonic system, given by the anti-symmetric stream function $\psi^a\left(x,y\right)$. According to the first and third parts of Eq. 22, Fig. 1c represents the mid-latitude anti-symmetric component of the meridional mass transport $M_y^a\left(x,\ 1/2\right)$, and, analogously, Fig. 1c represents the symmetric component $M_y^s\left(x,\ 1/2\right)$. Fig. 2 refers to the case $F\left(y\right)=\sin\left(\pi y\right)$, fit for an idealised subpolar gyre. Fig. 2a depicts a cyclonic system, while Fig. 2b is a couple cyclone, anti-cyclone with opposite signs with respect to the case 1. Openly, the stream lines are the same in both the cases 1 and 2. The resulting components of mass transport $M_y^a\left(x,\ 1/2\right)$ and $M_y^s\left(x,\ 1/2\right)$ have opposite signs with respect to the case 1, as shown in Figs. 2c and 2d, respectively. The key point, see Fig. 3a, relies on the *constant* sign of the product $M_y^s M_y^a$ in each of the half domains

$$Q_1=\left\{\left(x,y\right):\ -1\le x\le 0,\ 0\le y\le 1\right\},\ Q_2=\left\{\left(x,y\right):\ 0\le x\le 1,\ 0\le y\le 1\right\}$$

regardless of the sign of $F\left(y\right)$, thus in accordance with inequality 45. Finally, Fig. 3b plots $M_y^2\left(x,\ 1/2\right)$ in the full longitudinal range, in accordance with inequality 47.

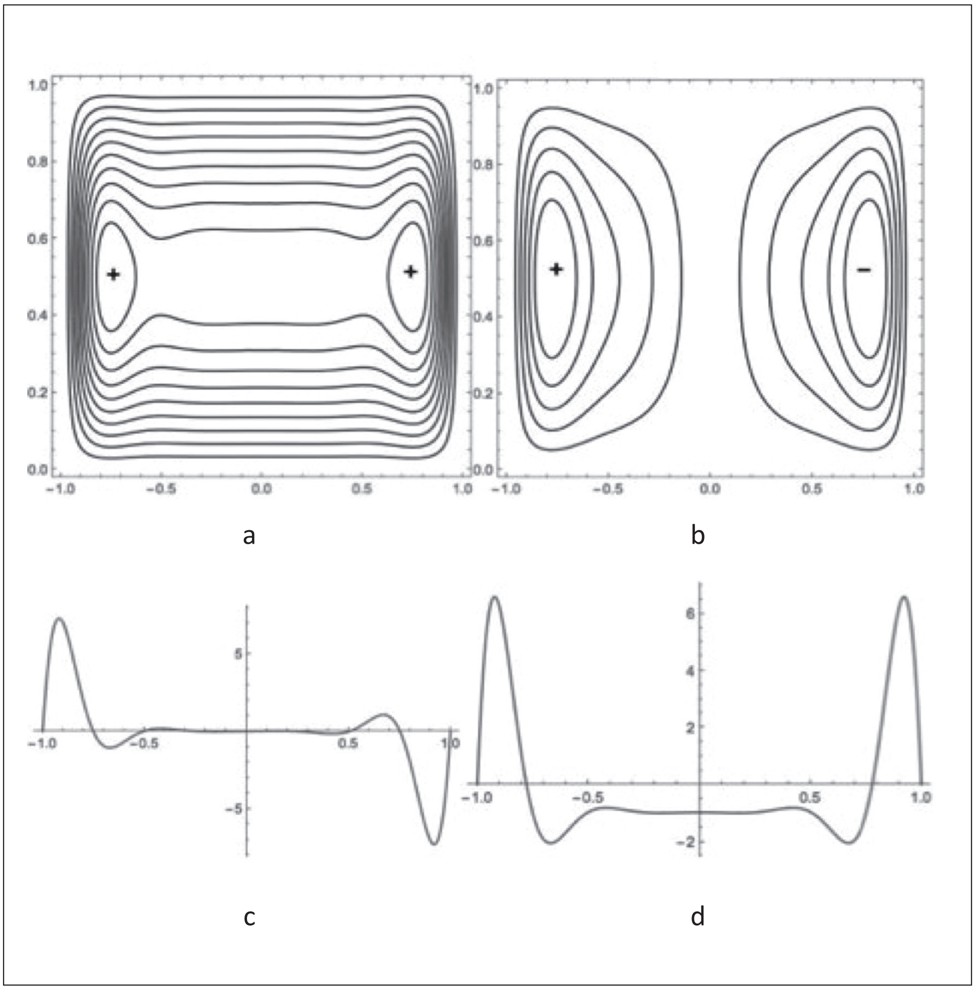

Fig. 1 - Streamlines of $\psi^s(x,y)$ in the case $F(y)=-\sin(\pi y)$ (a). Streamlines of $\psi^a(x,y)$ for the same forcing as above (b). Plot of the mid basin anti-symmetric transport $M_y^a(x,\ 1/2)=(\partial\psi^s/\partial x)_{y=1/2}$ as a function of longitude (c). Plot of the mid basin symmetric transport $M_y^s(x,\ 1/2)=(\partial\psi^a/\partial x)_{y=1/2}$ as a function of longitude (d).

5. Conclusions

The conclusions drawn from the model analysed in this investigation are twofold:

- westward intensification involves the dynamics of the flow in the full basin, via the anti symmetric part of the transport stream function. Its evidence is limited to a narrow region of the basin because of the peculiar superposition of the symmetric and anti symmetric components of the transport, as described in sub section 4.2;
- westward intensification is an *a priori* property of the solutions of the model, regardless of the shape of the forcing (Eq. 7).

The same method can be applied to a bottom dissipated system, with the term $-r\nabla^2\psi$ in place of $A_h\nabla^4\psi$. In this case, the basic Eqs. 26 and 28 are substituted by:

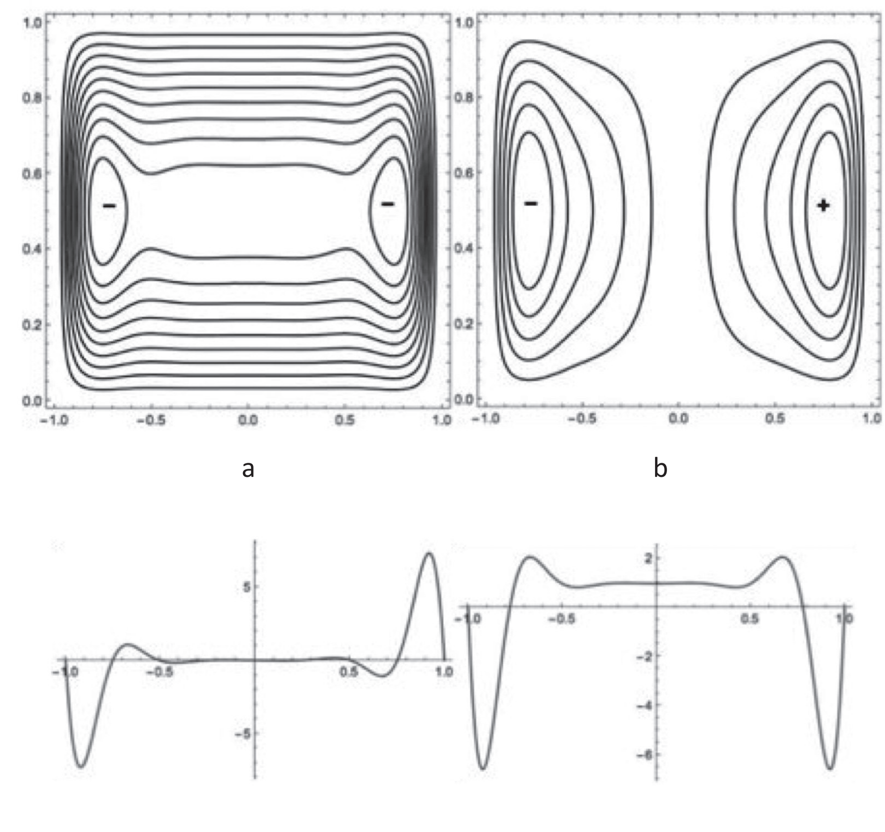

Fig. 2 - Streamlines of $\psi^s(x,y)$ in the case $F(y)=\sin(\pi y)$ (a). Streamlines of $\psi^a(x,y)$ for the same forcing as above (b). Plot of the mid basin anti-symmetric transport $M_y^a(x,\ 1/2)=\left(\partial\psi^s/\partial x\right)_{y=1/2}$ as a function of longitude (c). Plot of the mid basin symmetric transport $M_y^s(x,\ 1/2)=\left(\partial\psi^a/\partial x\right)_{y=1/2}$ as a function of longitude (d).

$$\beta\frac{\partial\psi^s}{\partial x}=-r\nabla^2\psi^a$$
$$\beta M_y^s M_y^a=-r\frac{\partial\psi^a}{\partial x}\nabla^2\psi^a \qquad (59)$$

respectively. Conclusions drawn from Eq. 60 are exactly the same as those listed above. The decomposition of fields in symmetric and anti-symmetric terms has been successfully used also in some previous investigations, for instance in Badin *et al*. (2009) and in Crisciani and Badin (2014).

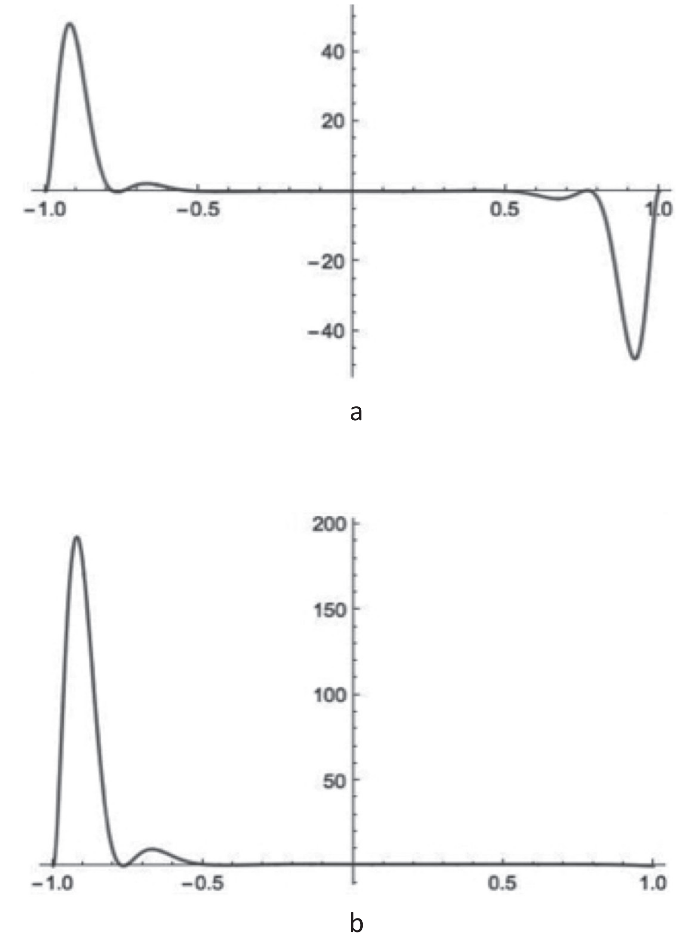

Fig. 3 - Plot of the product $M_y^s(x,\ 1/2)M_y^a(x,\ 1/2)$ as a function of longitude (a). The plot is independent of the sign of the sinusoidal forcing. Plot of transport square $M_y^2(x,\ 1/2)$ as a function of longitude (b).

REFERENCES

Badin G., Cavallini F. and Crisciani F.; 2009: *Zonally-aligned gyre solutions of linear models of wind-driven ocean circulation*. Nuovo Cimento B, 124, 653-669.

Cavallini F. and Crisciani F.; 2013: *Quasi-geostrophic theory of oceans and atmosphere*. Springer Verlag, Berlin, Germany, 385 pp.

Crisciani F. and Badin G.; 2014: *A note on the symmetric and anti-symmetric constituents od weakly-non linear solutions of classical wind-driven ocean circulation models*. Eur. Phys. J. Plus, 129, 142 .

Crisciani F. and Purini R.; 1997: *Boundary condition of Sverdrup solution from flow energetics*. J. Phys. Oceanogr., 27, 357-361.

Cushman-Roisin B.; 1994: *Introduction to geophysical fluid dynamics*. Prentice Hall, Englewood Cliffs, U.S.A., 320 pp.

Munk W.; 1950: *On the wind-driven ocean circulation*. J. Meteorolog., 7, 79-93.

Pedlosky J.; 1965: *A note on the western intensification of oceanic circulation*. J. Mar. Res., 23, 207-209.

Pedlosky J.; 1987: *Geophysical fluid dynamics*. Springer Verlag, Berlin, Germany, 710 pp.

Pedlosky J.; 1996: *Ocean circulation theory*. Springer Verlag, Berlin, Germany, 453 pp.

Stommel H.; 1948: *The westward intensification of wind-driven ocean currents*. Trans. Am. Geophys. Union, 29, 202-206.

Stommel H.; 1958: *The Gulf Stream, a physical and dynamical description*. University of California Press, Berkeley, U.S.A., 202 pp.

Vallis G.K,; 2006: *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, Cambridge, England, 745 pp.

Welander P.; 1959: *On the vertically integrated mass transport in the ocean*. In: Bolin B. (ed), The atmosphere and sea in motion, Rockfeller Institute Press, New York, U.S.A., pp. 75-100.

Wunsch C.; 2011: *The decadal mean ocean circulation and Sverdrup balance*. J. Mar. Res., 69, 417-434.

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