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How to compute orthometric altitudes

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Abstract - In countries with a very rugged topography, the orthometric altitudes are better adapted than other scientific altitudes, particularly for the interpolation of the geoid between computation points. The computation of this kind of altitude is, however, complicated because of the unknown mean gravity values along the vertical. We propose a method for computing the mean gravity value along the vertical, joining measurement points and the geoid allowing us to compute the orthometric altitude in a very simple way. This method is based on the prolongation of gravity anomalies between general surfaces. We present a test profile for the Swiss Alps which shows an accuracy better than 0.5 mGal in the determination of g_{mean}.

1. Introduction

Geoidal ondulations are often used for the interpretation of the deep structure of the Earth's crust at wavelengths greater than 50 km. In order to use these kind of data well they have to be easily interpolated in the form of a regular grid, well suited for inversion algorithm. An accurate interpolation of the geoidal undulations can be performed when the computed geoid is relatively smooth. In countries with a very rugged topography the orthometric altitudes are the best suited for producing a smooth and well interpolated geoid.

Altitude zero is defined by the geoid being the equipotential surface represented by the mean sea level. The altitudes have two different definitions: one is purely geometric and represents the vertical length between the point considered and the corresponding point on the geoid; the second one is defined by the distance between two equipotential surfaces one being the geoid. It is clear from these definitions that two points having the same physical altitude do not automatically have the same geometrical altitude. This is due to the fact that the gravity field of the Earth is not formed by equidistant equipotential surfaces.

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Let's suppose two points A_1 and B_1 are located on an equipotential surface which is not parallel to the geoid: they have the same physical altitude but obviously not the same geometrical one. To establish the altitudes the surveyor determines, sequentially, the height difference between the stations along a path joining a reference point to the point to be measured. The altitude at the end point is then the sum of the differences of the geometrical heights between adjacent points. Because this sum is not uniform (in a mathematical sense) the measurements carried out in one direction do not give the same results as those measured in the other direction. This can be avoided by replacing the geometrical altitudes by the so called geopotential quotes. For this, a gravity campaign has to be carried out in parallel with the levelling campaign.

The potential W_B of a point B with respect to a point A can be computed by

$$W_B = W_A - \int_A^B g\, dh. \tag{1}$$

By setting $W_A = 0$ and changing the sign of the sum one gets the geopotential quote.

The computation of the geopotential quote of a point B is straightforward:

$$C = W_{geod} - W_B = \sum_i g_i\, dh_i, \tag{2}$$

where:

g_i are the values of gravity at the levelling stations,

dh_i, are the altitude differences between the leveling stations up to point B,

W_{geod}, is the gravity potential on the geoid.

This value, contrary to the classical altitudes obtained by levelling, is independent of the path taken for the measurements between A and B.

However, the units are difficult to integrate in the classical notion of altitude. In order to avoid this difficulty the geodesists have defined different altitude systems which are compromises between geometrical and physical height systems. These altitudes are called scientific altitudes. One of these is the orthometric altitude and is defined as:

$$h_0 = \frac{\Sigma_i\, g_i\, dh_i}{g}, \tag{3}$$

where $\overline{g}$ is the mean value of g along the true vertical joining the measurement point and the corresponding point on the geoid given by,

$$\overline{g} = \frac{1}{h} \int_0^P g\, dh. \tag{4}$$

Orthometric altitudes are well suited in countries with a very rugged topography. Contrary to normal altitudes, they allow us to produce a very smooth geoid which can be interpolated between the computation points.

The computation of the orthometric altitude can be conducted in different ways. The most basic formulation shows that the knowledge of the geopotential quote is first necessary, by then dividing this quantity by the mean value of the gravity along the vertical, joining the point B and the geoid, leads to the orthometric height:

$$H_{ortho} = \frac{C}{\overline{g}} . \tag{5}$$

In this computation the difficulty is obviously the determination of $\overline{g}$.

Another way to compute the orthometric altitude is to apply a correction to the geometrical altitude

$$\begin{aligned} H_B &= H_A + \Delta H_{AB}, \\ \Delta H_{AB} &= \Sigma_i \, dh_i = \varepsilon_{AB}, \end{aligned} \tag{6}$$

where ε_{AB} = orthometric correction = term also containing $\overline{g}$, is given after Heiskanen and Moritz (1967) by:

$$\varepsilon_{AB} = \frac{1}{g_0} \sum_{A}^{B} (g_i - g_0) \, dh_i + H_A \left(\frac{\overline{g}_A - g_0}{g_0} \right) - H_B \left(\frac{\overline{g}_B - g_0}{g_0} \right), \tag{7}$$

and

$$\overline{g} = g_{obs} + 0.5 \, g_{FA} - g_{Boug} - g_{Top} + \frac{1}{H} \int_0^P g_{Top} \, dh, \tag{8}$$

with:

g_{obs} = measured value of g,
g_{FA} = value of the free air correction = $0.3086 \times h$,
g_{Boug} = value of the simple Bouguer correction = $2\pi G\rho h$,
g_{Top} = value of the gravitational effect of the topography,
g_0 = theoretical value of g on the ellipsoid at latitude A resp. B,

or after Wirth (1990),

$$\overline{g} = g_{obs} + 0.5 \, g_{FA} - g_{Boug} - g_M - \frac{1}{H} (T_M^P - T_M^0), \tag{9}$$

where

g_M = effect of the whole masses on g,
T_M^i = effect of the masses on the potential, at the point i.

Heiskanen and Moritz (1967, chapter 4, p. 167) describe, in detail, the classical methodology for computing orthometric altitudes when the topography is relatively smooth. They also describe how to proceed when one wishes to continue the anomaly downward to the sea level or to any surface in-between (chapter 8, pp. 317 and 318). This continuation allows the computation of the theoretical values of g on different surfaces between the geoid and the topography and consequently, the possibility to calculate the mean value $\bar{g}$. This methodology, although elegant, is not easy to apply because it needs to solve an integral equation of the first kind. In the formulation of Wirth (1990), it is clear that in the calculations of g_M as well as for T_M^P and T_M^0 the densities and also the limits of the disturbing bodies have to be known with sufficient accuracy in order to reach a reasonable precision.

The determination of densities is not a difficult task for outcropping rocks. For deep seated bodies this determination depends on the knowledge of the seismic velocities V_p and on a hypothetical V_p-density relationship. Apart from the uncertainty of the densities the determination of the limits between geological units can be a real problem at depth.

In order to overcome this problem, as well as the problem of solving an integral equation, we propose another method for computing the orthometric altitude. This method requires only the computation of the mean value of g along the vertical.

2. Another method

2.1. Introduction

Suppose that one would be able to compute the Bouguer anomaly on different surfaces located between the geoid and the measurement point:

$$\Delta g_{Boug}(z) = g_{obs}(z) - [g_0(z) - g_{FA}(z) + [g_{Boug}(z) - \Delta g_{top}(z)]. \tag{10}$$

Then the computation of g on the same surfaces will be very simple and given by:

$$g_{obs}(z) = \Delta g_{Boug}(z) + [g_0(z) - g_{FA}(z) + g_{Boug}(z) - \Delta g_{top}(z)]. \tag{11}$$

where

Δg_{Boug} = full Bouguer anomaly,
g_{FA} = value of the free air correction (e.g. $0.3086 \times h$,
g_{Boug} = effect of the layer of rocks between the ellipsoid and the point of computation,
Δg_{top} = effect of the material located above the surface at altitude z,
g_0 = theoretical value of g at the latitude of the point of computation, and at altitude z.

From these values it would be possible to compute the mean value along the vertical by numerical integration. In fact, the problem of determining the mean value of g by this method is only resumed by the calculation of the Bouguer anomaly at depth. This can be done by the so-called vertical continuation of potential field techniques.

The advantage of this method, apart from the difficulty to compute the Bouguer anomaly at depth, is that no geological models of any kind are necessary, they are already included in the Bouguer anomaly. The inaccuracy introduced by the uncertainties on densities and limits of the disturbing bodies are automatically eliminated.

2.2. The vertical continuation of the Bouguer anomaly

Many methods have been proposed to carry out the vertical continuation of potential field data. Some are based on Fourier transformation techniques some others on the approximation of the anomaly by a finite Fourier series.

A very elegant method to carry out the continuation of gravity anomalies is based on the property of multiplicity of the solution of the gravity field which was given by Dampney (1969). This author pointed out that the value of an anomaly Δg at any point with coordinates x, y, z, located outside the disturbing body can be written as:

$$\Delta g\,(x, y, z) = G \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma\,(\alpha, \beta, h)\,(z - h)}{[(x - \alpha)^2 + (y - \beta)^2 + (z - h)^2]^{\frac{3}{2}}}\, d\alpha\, d\beta, \tag{12}$$

where $\sigma\,(\alpha, \beta, h)$ is a density contrast and G the universal gravitational constant. This equation shows that for a given anomaly it is possible to compute a density or mass distribution on any surface, which gives a gravity effect equal to that produced by the real disturbing body. Following this, it is possible to use this density or mass distribution to recompute the anomaly to any desired surfaces located outside the disturbing sources.

In his technique Dampney (1969) used a horizontal plane as the support surface for the equivalent source and showed that the anomaly can be synthetizied with the help of a discontinuous and finite distribution of masses located on this plane.

Calling g_i the value of the anomaly at points i and m_i the elementary mass located on the support surface just below point i one can write, using the superposition property of potential fields:

$$\begin{aligned}
g_1 &= a_{11} m_1 + a_{12} m_2 + \ldots + a_{1k} m_k + \ldots + a_{1M} m_M \\
g_2 &= a_{21} m_1 + a_{22} m_2 + \ldots + a_{2k} m_k + \ldots + a_{2M} m_M \\
&\vdots \\
g_N &= a_{N1} m_1 + a_{N2} m_2 + \ldots + a_{Nk} m_k + \ldots + a_{NM} m_M,
\end{aligned}$$

and $M = N$ where
M = number of masses,
N = number of measurement points.

The coefficients a_{ik} are calculated by:

$$a_{ik} = \frac{G\,(z_i - h)}{[(x_i - \alpha_k)^2 + (y_i - \beta_k)^2 + (z_i - h)^2]^{\frac{3}{2}}}\,. \tag{13}$$

This leads to a linear system of N equations of M unknowns which in principle allows the computation of the masses m_k.

Starting from the same idea as Dampney (1969), Graber-Brunner et al. (1991), Klingelé (1997) developed an upward continuation technique for data in two dimensions. Instead of having a mass distribution on a horizontal surface, the equivalent layer is parallel and below the topography and formed by blocks of different densities. Each block produces a gravity effect g_i on each measurement point equal to the product of its density by its effect computed for a density equal to one. It follows that:

$$g_1 = s_1\,g^*_{11} + s_2\,g^*_{12} + \ldots + s_k\,g^*_{1k} + \ldots + s_N\,g^*_{1N}$$

$$\vdots$$

$$g_j = s_1\,g^*_{j1} + s_2\,g^*_{j2} + \ldots + s_k\,g^*_{jk} + \ldots + s_N\,g^*_{jN}$$

$$\vdots$$

$$g_N = s_1\,g^*_{N1} + s_2\,g^*_{N2} + \ldots + s_k\,g^*_{Nk} + \ldots + s_N\,g^*_{NN},$$

with g^*_{ik} = effect of the block k on point i, and s_k the density of the block k. It is easy to see that in this case the system has to be solved for the densities s_k instead of for the masses m_k. The finite geometry of the layer and the sizes of the elementary blocks lead to a very stable linear system which can be solved easily by any standard matrix method.

2.3. Practical aspects of the computation

For the calculation of the second part of the right hand side of formula (10)

$$\Delta g_{Boug} = g_{obs} - [g_0 - g_{FA} + g_{Boug} - \Delta g_{top}], \tag{14}$$

it is preferable from a practical point of view to compute only two terms. The first one is given by the theoretical value of g at the coordinates x,y and altitude z, and computed, for example, by means of the WGS84 formula. The second term is the total effect of the masses located between the ellipsoid and the topography. This can be carried out very simply by any numerical integration method:

$$\Delta g = g_{mes} - [g_{th}\,(\varphi, z) - \Delta g_{TopoMasses}]. \tag{15}$$

2.4. Test of the method

In order to test the validity of the method a synthetic example was used. This example is based on a real situation encountered in the Swiss Alps, more precisely along a profile joining the city of Göschenen (UR) and the Gotthard Pass and ending two kilometers to the west of the city of Airolo (TI) (Fig. 1).

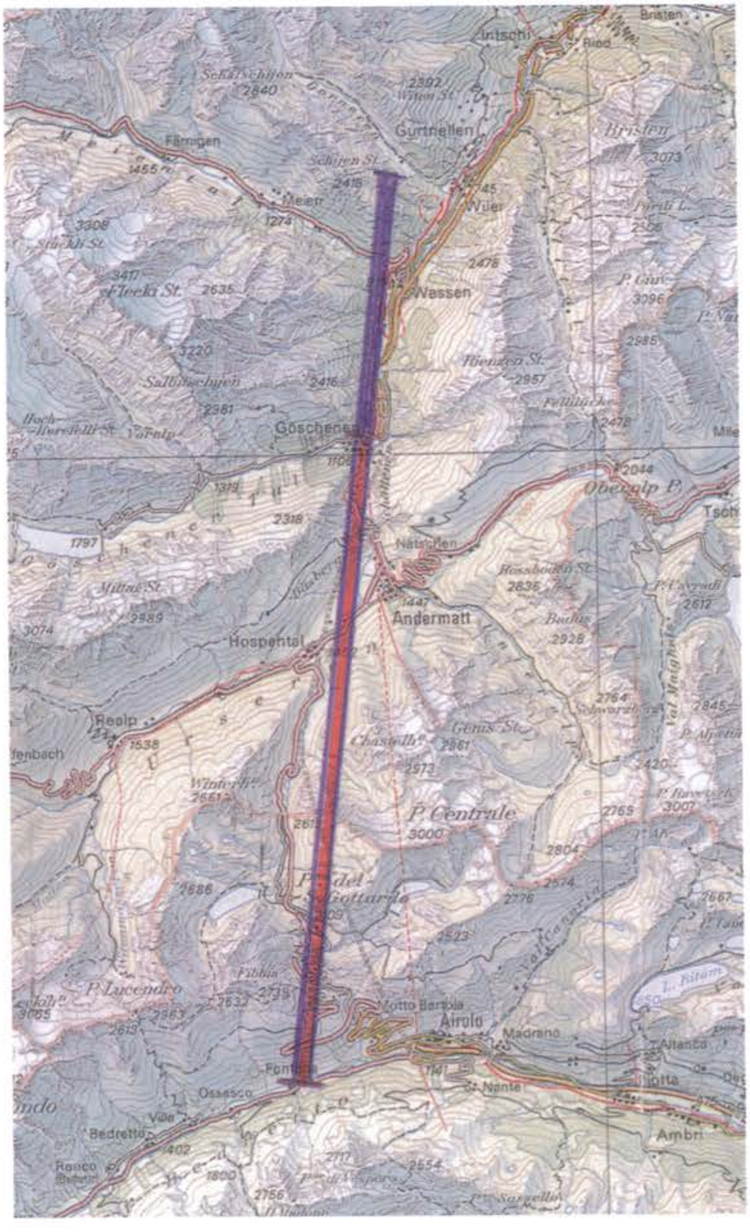

Fig. 1 - Geographical location of the test profile. The blue and the red segments represent the part of the Swiss gravity map digitized for producing the synthetic anomaly. The red segment corresponds to the part of the profile used for the determination of the mean *g* value. The topography and the synthetic Bouguer anomaly also correspond to this part of the profile.

Along this profile the topography has been approximated by a series of straight lines joining points with altitudes extracted from the topographic maps of Switzerland at 1:500,000 scale (Fig. 2).

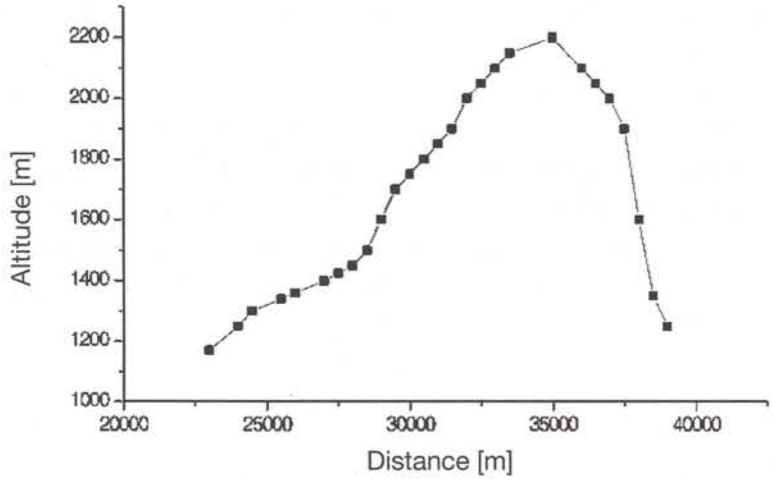

Fig. 2 - Profile showing the simplified topography along the red part of the profile of Fig. 1. The black dots represent the gravity stations. The origin of the profile is arbitrarily fixed.

Along the same profile, the Bouguer gravity anomaly has been extracted from the Bouguer gravity map of Switzerland, at 1:500,000 scale, on stations corresponding to some (not all) of the points defining the topography (Fig. 2).

From the real anomaly the shape and the density of a test structure was determined experimentally by means of a trial and error method. The anomaly produced by this structure shows good similarities with the real anomaly (Fig. 3).

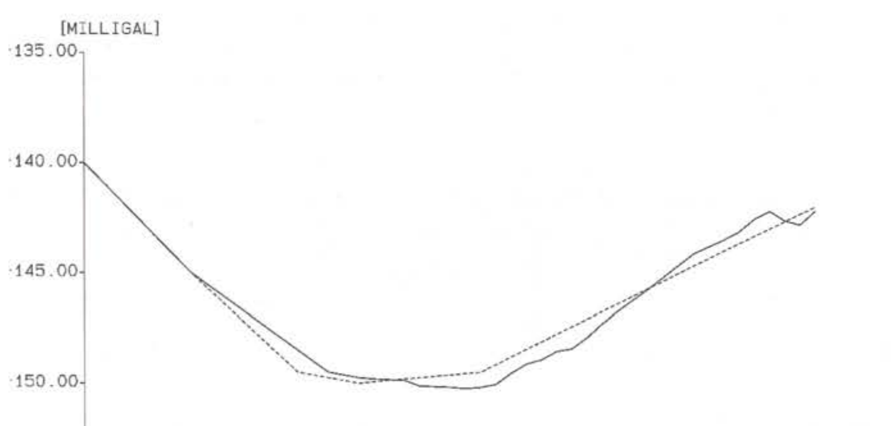

Fig. 3 - Synthetic Bouguer anomaly (continuous line) and measured Bouguer anomaly along the full profile of Fig. 1 (blue and red segments). The origin of the profile is arbitrarily fixed.

This structure allowed us to compute the Bouguer anomaly at different levels and then to determine the mean *g* value along the vertical for any point along the profile by means of a numerical integration. The anomaly was computed by using the test structure and the synthetic topography.

The altitudes and the synthetic Bouguer anomaly at the stations along the profile, as well as the synthetic topography, were then used as input for the computation of the Bouguer anomaly at different levels by means of the equivalent source technique previously described. From the Bouguer anomalies obtained at different levels the *g* values were computed by means of formula (15) and then numerically integrated along the vertical, from the ellipsoid to the altitude at 2200 m. The constant altitude of 2200 m was chosen first by to simplify the computation and secondly to give a better comparison, every vertical having the same integration interval.

The results of this computation are shown in figure 4 together with the exact solutions obtained from the test structure.

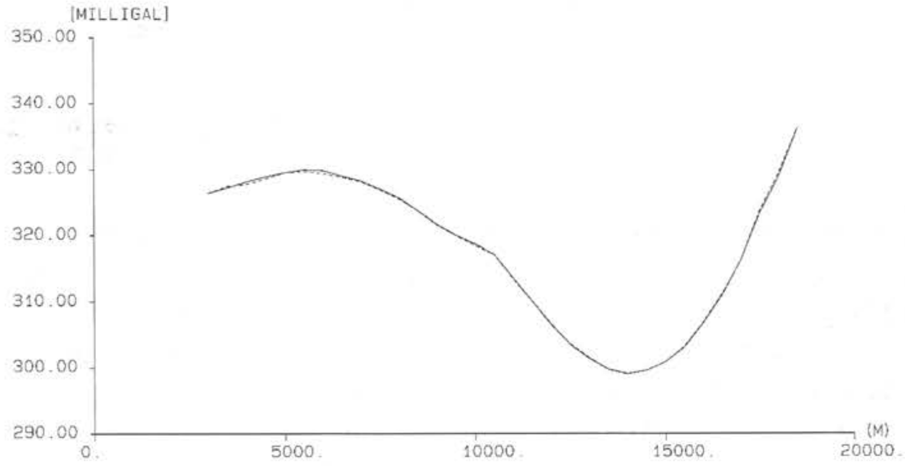

Fig. 4 - Mean value of *g* along the vertical between the ellipsoid and the altitude of 2200 m for the profile of Fig. 1. The origin of the profile is arbitrarily fixed. A value of 980 000 mGal has been subtracted from the real value for reason of clarity. Dotted line: values computed with the disturbing structure (exact solution). Continuous line: values computed by means of the equivalent layer technique.

The differences between these two sets of data never exceed 0.50 mGal with a mean value of -0.04 mGal (Table 1).

3. Conclusions

The method presented in this paper clearly shows that the orthometric corrections to be applied to the levelled altitudes can be accurately computed without any knowledge of the geology of the measured area. Neither the densities nor the limits of the disturbing bodies like the roots of the Alps or the Ivrea body have to be determined. Only the Bouguer anomaly must be known.

The technique presented can be extended to three dimensional cases without great effort.

Table 1 - Comparison of the true and the computed mean values of *g* along the verticals for the profile of Fig. 1.

Distance in meters	True value of the mean *g* value	Computed mean *g* value	Difference computed *g* - true *g*
23000.	980326.38	980326.38	0.00
23500.	980327.19	980327.44	0.25
24000.	980328.13	980327.75	-0.38
24500.	980328.88	980328.56	-0.32
25000.	980329.50	980329.50	0.00
25500.	980329.94	980329.69	-0.25
26000.	980329.81	980329.38	-0.42
26500.	980329.00	980328.81	-0.19
27000.	980328.25	980328.13	-0.12
27500.	980327.00	980326.88	-0.12
28000.	980325.69	980325.50	-0.19
28500.	980323.69	980323.69	0.00
29000.	980321.63	980321.50	-0.13
29500.	980320.00	980319.94	-0.06
30000.	980318.69	980318.44	-0.25
30500.	980317.19	980317.19	0.00
31000.	980313.69	980313.56	-0.13
31500.	980310.13	980310.13	0.00
32000.	980306.63	980306.50	-0.13
32500.	980303.50	980303.63	0.13
33000.	980301.38	980301.50	0.12
33500.	980299.75	980299.69	-0.06
34000.	980299.06	980299.00	-0.06
34500.	980299.56	980299.56	0.00
35000.	980300.81	980300.81	0.00
35500.	980303.06	980303.19	0.13
36000.	980306.69	980306.81	0.12
36500.	980310.94	980311.13	0.19
37000.	980316.25	980316.25	0.00
37500.	980323.31	980323.69	0.38
38000.	980328.88	980329.38	0.50
38500.	980336.19	980336.19	0.00

References

Dampney C. N. G.; 1969: *The equivalent source technique*. Geophysics, **34**, 1, 39-53.

Graber-Brunner V., Klingelé E. E. and Marson I.; 1991: *An improved solution for the problem of upward continuation of gravity data in rugged topography.* Boll. Geof. Teor. Appl., **33**, pp. 135-144.

Heiskanen W. A. and Moritz H.; 1967: *Physical Geodesy.* W. H. Freeman, San Francisco. 364 pages.

Klingelé E. and Olivier R.; 1978: *Carte gravimétrique de la Suisse - Anomalie de Bouguer - 1/500'000^e.* Service Topographique Fédéral, Wabern.

Klingelé E.; 1997: *2-D gravimetric study of the crystalline basement of the Rawyl depression.* In: Deep structure of the Swiss Alpes, results of NRP 20. O.E. Pfiffner, (ed) Birkhäuser Verlag pp. 154-159.

Wirth B.; 1990: *Höhensysteme, Schwerepotentiale und Niveauflächen: Systematische Untersuchungen zur zukünftigen terrestrischen und GPS-gestützten Höhenbestimmung in der Schweiz.* Geodätisch-geophysikalische Arbeiten in der Schweiz, Band 42, 203 Seiten.