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A construction of Galerkin's matrix in quasi-geoid determination

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Abstract. In this paper we discuss the use of variational methods for the determination of the disturbing potential as a solution of the geodetic boundaryvalue problem. The emphasis is on the interpretation in terms of function bases and the construction of elements in Galerkin's matrix. An approximation of these elements is shown and its accuracy examined for a system of base functions given by elementary potentials of Laplace's equation. It is demonstrated how the accuracy depends on the telluroid's topography and the depth of individual mass concentration points.

1. Introduction

The determination of the disturbing potential T from surface gravity data is usually treated as a classical solution of the geodetic boundary value problem. (T is a smooth function which satisfies Laplace's equation and the respective boundary condition point-wise.) In this paper, on the contrary, T is defined by means of an integral identity connected with the boundary value problem in question. This is the basic idea of variational methods, see Nečas (1967) or Rektorys (1974). T as defined above represents a generalization of the classical solution, see also Holota (1996, 1997a, 1997b, 1998a, 1998b, 1999).

For our solution domain Ω given by the exterior of the telluroid we consider Sobolev's weight space $W_2^{(1)}(\Omega)$ endowed with inner product

$$(u,v)_1 \equiv \int_\Omega \frac{uv}{|\boldsymbol{x}|^2} d\boldsymbol{x} + \sum_{i=1}^{3} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} d\boldsymbol{x}, \tag{1}$$

where x_i , $i = 1, 2, 3$, are rectangular Cartesian coordinates in Euclidean 3-dimensional space $\boldsymbol{R}^3$. The boundary $\partial\Omega$ of Ω is supposed to have a certain degree of regularity. Putting $\Omega' = \boldsymbol{R}^3 - \Omega \cup \partial\Omega$,

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we assume that *Ω'* is the domain with Lipschitz' boundary.

Now we define the solution of our problem as a function $T \in W_2^{(1)}$ such that

$$((T, v)) = \int_{\partial\Omega} vfdS \tag{2}$$

holds for all $v \in W_2^{(1)}$, provided that $((u, v))$ is a bilinear form connected with the geodetic boundary value problem and *f* is a square integrable function on $\partial\Omega$.

The geodetic boundary value problem is an oblique derivative problem and the respective bilinear form has been discussed in Holota (1996, 1997a, 1998a, 1999). It turned out that a modification is more convenient. We first denote, by *U,* the potential of the model (normal) gravity γ, $\gamma = |\boldsymbol{grad}\ U|$ and then put $((u, v)) = A(u, v) + a(u, v)$, where

$$A(u,v) = \sum_{i=1}^{3} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} d\boldsymbol{x} \quad \text{and} \quad a(u,v) = \int_{\partial\Omega} \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} uv \cos(\boldsymbol{h}, \boldsymbol{n}) dS \tag{3}$$

with $\boldsymbol{n}$ being the (unit) outer normal of $\partial\Omega$. and $\boldsymbol{h}$ denoting the direction of the outward normal to the level surface of the model gravity potential. As regards *f* , we put

$$f = f^* \cos(\boldsymbol{h}, \boldsymbol{n}), \;\; f^* = \Delta g + \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \Delta W - \frac{\partial T}{\partial t} \tan(\boldsymbol{h}, \boldsymbol{n}). \tag{4}$$

Here ΔW and Δg stand for the potential and the gravity anomaly, respectively and $\partial T/\partial t$ (referred to $\partial\Omega$.) is a component of $\boldsymbol{grad}\ \boldsymbol{T}$ in the tangential plane of an equi-potential surface of *U* in the direction of maximum inclination of the telluroid's topography.

It is well-known from (Molodensky et al., 1960, eq. V.5.22) or (Heiskanen and Moritz, 1967, eq. 8-21) that $(\partial T/\partial t)\ tan\ (\boldsymbol{h}, \boldsymbol{n}) = \gamma\ (\xi\ tan\ \beta_1 + \eta\ tan\ \beta_2\ )$, where ξ and η are the Molodensky defined components of the detection of the vertical and β_1 , β_2 are the angles of the slopes of the north-south and east-west telluroid profiles. Since β_1 and β_2 are usually small, approximate values of ξ and η are usually sufficient.

Finally, in our formulation *T* is defined as an element among an excessively great multitude of functions. It can be shown that it is enough to consider a space $H_2^{(1)}(\Omega)$ of those functions from $W_2^{(1)}(\Omega)$ which are harmonic in Ω and to reformulate our definition, i.e. to look for $T \in H_2^{(1)}(\Omega)$ such that Eq. (2) holds for all $v \in H_2^{(1)}(\Omega)$, see Holota (1998a, 1999).

2. Galerkin's Matrix

Taking into consideration the practice applied in geodesy, we agree that it is convenient to use Runge's property of Laplace's equation and to work with a space of functions which are harmonic outside a domain *B* completely embedded in the telluroid. Putting $B' = \boldsymbol{R}^3 - B$, we will denote this space by $H(B')$.

Following Neyman (1979) and an analogue to his reasoning related to the so-called Bjerhammar sphere, one can show that in terms of the norm $||u||_1 \equiv (u,u)_1^{1/2}$ the space $H(B')$ is dense in $H_2^{(1)}(\Omega)$. This enables an approximation of T by means of

$$T^{(n)} = \sum_{i=0}^{n} c_i^{(n)} v_i\,, \tag{5}$$

where $c_i^{(n)}$ are numerical coefficients and v_i are members of a function base of $H(B')$. Moreover, Eq. (2) offers a natural starting point for a numerical interpretation of the problem. Indeed, for the coefficients $c_i^{(n)}$ we can immediately write Galerkin's system

$$\sum_{i=0}^{n} c_i^{(n)}((v_i, v_j)) = \int_{\partial\Omega} v_j f^* \cos(h,n)\,dS, \tag{6}$$

where $j = 0,..., n$. Note that the boundary value problem under consideration is close to Stokes' problem. Thus system (6) is either weakly conditioned or even singular. A trick from (Hörmander, 1975) may be used as a way out, see also Holota, (1996).

In this paper we take for v_i, $i = 0,..., n$ a set of elementary potentials

$$v_i(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x} - \boldsymbol{y}_i|}, \quad \boldsymbol{y}_i \in B. \tag{7}$$

Our aim is to approach the computation of the elements $((v_i, v_j))$. Assume that the domain Ω' is star-shaped at the origin and define $\boldsymbol{y}_i$' as a point of intersection of $\partial\Omega$. with a radial ray passing through the point $\boldsymbol{y}_i$. For $i, j = 0,..., n$ put $R_{ij} = (|\boldsymbol{y}_i'| + |\boldsymbol{y}_j'|)/2$ or $R_{ij} = \max_{i,j} [\,|\boldsymbol{y}_i'|, |\boldsymbol{y}_j'|\,]$ in case that $(|\boldsymbol{y}_i'| + |\boldsymbol{y}_j'|)/2 \leq max_{i,j} [\,|\boldsymbol{y}_i|, |\boldsymbol{y}_j|\,]$. Now, writing R instead of R_{ij} for simplicity, we try to approximate $((v_i, v_i))$ by means of $((v_i, v_j))_R = A_R(v_i, v_i) + a_R(v_i, v_j)$, where

$$A_R(v_i, v_j) = \sum_{k=1}^{3} \int_{S_R} \frac{\partial v_i}{\partial x_k} \frac{\partial v_j}{\partial x_k} d\boldsymbol{x}, \quad a_R(v_i, v_j) = -\frac{2}{R} \int_{\partial S_R} v_i v_j\, dS \tag{8}$$

and $S_R \equiv \{\boldsymbol{x} \in R^3;\ |\boldsymbol{x}| > R\}$. ($R$ can be also defined in an average sense so that in a surrounding of $\boldsymbol{y}_i$' and $\boldsymbol{y}_j$' the sphere ∂S_R approximates the topography of the telluroid.) The diagonal terms can be computed by means of a standard integration. We obtain

$$((v_i, v_i))_R = \pi\left(\frac{2R}{R^2 - |\boldsymbol{y}_i|^2} - \frac{3}{|\boldsymbol{y}_i|}\ln\frac{R + |\boldsymbol{y}_i|}{R - |\boldsymbol{y}_i|}\right). \tag{9}$$

(As an example take $R = 6378\ km$ and $R - |\boldsymbol{y}_i| = 20\ km$. Then $((v_i, v_i))_R = 0.157\ km^{-1}$.)

For the off-diagonal terms the computation is a bit more complicated. It can be found in Holota (1999). The result is as follows:

$$((v_i,v_j))_R = \frac{2\pi}{R}\left[\frac{1}{L} - \frac{3}{2z}\ln\frac{L+z-\cos\psi_{ij}}{1-\cos\psi_{ij}} - 3\delta S\right], \tag{10}$$

$$L = \sqrt{1-2z\cos\psi_{ij}+z^2}, \quad z = \frac{|\boldsymbol{y}_i|\,|\boldsymbol{y}_i|}{R^2}, \quad \delta S = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2n^2+3n+1}z^n P_n(\cos\psi_{ij}) \tag{11}$$

and ψ_{ij} is the angle between the vectors $\boldsymbol{y}_i$ and $\boldsymbol{y}_j$. Note that in Eqs. (11) the magnitude of individual terms of the series for the computation of ∂S decreases very quickly. Therefore, it is enough to sum up its first terms only to guarantee sufficient accuracy.

3. Accuracy of an approximate element

The problem now is to examine the difference $d_{ij} = ((v_i, v_j)) - ((v_i, v_j))_R$. Putting $D' = \Omega - S_R$, $D'' = S_R - \Omega$ and $D = D' \cup D''$, we first define

$$M_1 = \frac{\max}{D}\left[\left|\frac{1}{\gamma}\frac{\partial\gamma}{\partial h}\right|\right], \quad M_2 = \frac{\max}{D}\left[\left|\frac{\partial}{\partial|\boldsymbol{x}|}\left(\frac{1}{\gamma}\frac{\partial\gamma}{\partial h}\right)\right|\right], \quad R_{\min} = \frac{\min}{D}\left[|\boldsymbol{x}|\right], \tag{12}$$

$$M_3 = \frac{\max}{\partial\Omega}\left[\left|\cos(\boldsymbol{h},\boldsymbol{n}) - \cos(\boldsymbol{x},\boldsymbol{n})\right|\right] \text{ and } M_4 = \frac{\max}{\partial S_R}\left[\left|\frac{1}{\gamma}\frac{\partial\gamma}{\partial h} + \frac{2}{R}\right|\right]. \tag{13}$$

Then, referring to Holota (1998b), we can write immediately that

$$\begin{aligned} |d_{ij}| \le {} & \left(\frac{4M_1}{R_{\min}} + M_2\right)\int_D v_i v_j d\boldsymbol{x} + M_1\int_D v_i|\operatorname{grad} v_j|d\boldsymbol{x} + M_1\int_D v_j|\operatorname{grad} v_i|d\boldsymbol{x} + \\ & + \int_D|\operatorname{grad} v_j|\;|\operatorname{grad} v_j|d\boldsymbol{x} + M_1 M_3\int_{\partial\Omega} v_i v_j dS + M_4\int_{\partial S_R} v_i v_j\, dS. \end{aligned} \tag{14}$$

In order to give an example let us consider the so-called simple Molodensky problem. In this case $\boldsymbol{h}/|\boldsymbol{h}| = \boldsymbol{x}/|\boldsymbol{x}|$ and $(1/\gamma)\,(\partial\gamma/\partial h) = -2/|\boldsymbol{x}|$ so that $M_1 = 2/R_{\min}$ and $M_2 = 2/R^2_{\min}$ while $M_3 = M_4 = 0$. Thus for v_i, $i = 0,..., n$, given by eq. (7), we have

$$|d_{ij}| \le \frac{10}{R^2_{\min}} J^{(11)}_{ij} + \frac{2}{R_{\min}} J^{(12)}_{ij} + \frac{2}{R_{\min}} J^{(21)}_{ij} + J^{(22)}_{ij}, \tag{15}$$

$$J^{(nm)}_{ij} = \int_D \frac{1}{|\boldsymbol{x}-\boldsymbol{y}_j|^n |\boldsymbol{x}-\boldsymbol{y}_j|^m} d\boldsymbol{x}, \quad n,m = 1,2. \tag{16}$$

(Here D is actually related to i and j through the definition of $R = R_{ij}$. Thus we should also write D_{ij}, but for simplicity we will omit this notation.)

The estimation of $J_{ij}^{(nm)}$ is a problem of a technical nature. Details are in Holota (1998b). We approach the accuracy of the diagonal terms first, i.e. we will discuss $|d_{ii}|$. The individual estimates contain a number of parameters which should be adapted to statistics of the topography of the Earth and to the depth of mass concentration points. For this purpose define $D_{int} \equiv \{\boldsymbol{x} \in D;\ 0 \le \psi \le \psi_0\}$ and $D_{ext} \equiv \{\boldsymbol{x} \in D;\ \psi_0 \le \psi \le \pi\}$,where ψ is the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}_i$. Thus for $\psi_0 \in (0, \pi)$ we can put

$$R_{\min}^{\text{int}} = \frac{\min}{D_{\text{int}}}[|\boldsymbol{x}|],\ R_{\max}^{\text{int}} = \frac{\max}{D_{\text{int}}}[|\boldsymbol{x}|],\ R_{\min}^{ext} = \frac{\min}{D_{ext}}[|\boldsymbol{x}|]\ \ and\ \ R_{\max}^{ext} = \frac{\max\ \vartheta}{D_{ext}}[|\boldsymbol{x}|]. \tag{17}$$

Choose now e.g. $\psi_0 = 2°$and suppose that $R_{\min}^{\text{int}} - |\boldsymbol{y}_i| = 20$ km. In addition assume as a mere illustration that $R_{\max}^{\text{int}}$-$R_{\min}^{\text{int}} = 0.2$ km and $R_{\max}^{\text{ext}}$ - $R_{\min}^{\text{ext}} = 20$ km and take approximately $R_{\min}^{\text{int}} \approx R_{\min}^{\text{ext}} \approx R_{\min} \approx 6356\ km$. Under these parameters we arrive at $|d_{ii}| \le 0.0033\ km^{-1}$. Comparing this estimate with $((v_i, v_i))_R$ obtained in the example in Sect. 2, we can see that for $R - |\boldsymbol{y}_i| = 20$ km and the topography characterized as above $((v_i, v_i))_R$ represents $((v_i, v_i))$ with a relative accuracy of 2.1%. Nevertheless we have good reasons to believe that in reality the accuracy is much better.

As regards the off-diagonal terms we can repeat, with some modifications, similar steps as above and show that their accuracy is of the same level as that of the diagonal terms.

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