Investigation of the Molodensky series terms for terrain reduced gravity field data

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Abstract. Masses associated with the local topography are a dominant source of short wavelength gravity field variations. In the modeling of the gravity field it is thus an advantage to eliminate the effect of the terrain in a remove-restore procedure. In this process, terrain reductions are computed for the observation stations at the Earth's surface, considering either the complete topography, the topography and its isostatic compensation, or the residual topography (RTM technique). Hence, the reduced gravity field observations are referring to the actual ground level. Therefore, Molodensky's theory, considering data on non-level surfaces, should be applied in the traditional gravity field modeling approaches, while in collocation this is handled directly through height dependent covariance functions. In this paper, we provide numerical examples for the computation of the Molodensky series terms associated with the traditionally unreduced observations as well as in connection with various terrain-reduced data. Due to the smoothing, resulting from the terrain reductions, the magnitude of the Molodensky terms is reduced as well and the series convergence is improved. The computations are based on the Fast Fourier Transform (FFT) technique. The numerical tests are done in a mountainous area of the European Alps. The terrain data are on grids with various grid spacings starting at a resolution of 200 m. One of the main goals of this study is to investigate the magnitude of the Molodensky terms with regard to the computation of an improved European quasigeoid model.

1. Introduction

Molodensky's theory takes the Earth's surface as the boundary surface and the corresponding solution consists of integrals involving gravity anomalies and topographic heights (Molodensky

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et al., 1962; Moritz, 1980). In practical evaluation procedures, the only requirement for both input data sources is to be available on the same grid. There are different solutions for the Molodensky boundary value problem available. Molodensky et al. (1962) gave a solution as a series of double integrals on the sphere by the use of a surface layer. Other investigators used a kind of linearization of the series (Brovar, 1964; Pellinen, 1972), or derived a solution by analytical continuation of the gravity anomalies to a level surface (Moritz, 1980; Sideris, 1987). Although some of the solutions are better suited for practical evaluation, they are all termwise equivalent in planar approximation (Li et al., 1995). The main objective of this paper is to provide numerical results for height anomaly predictions based on Molodensky's solution in connection with different terrain reductions for the gravity data. The numerical tests are done in a mountainous area of the Alps. The magnitude of the Molodensky series terms to the 3rd order is studied for different terrain-reduced data sets and comparisons of the different solutions with GPS/leveling data are given. From these results some interesting conclusions are derived for the computation of a new European quasigeoid model.

2. Remove-Restore technique

The introduction of FFT techniques in gravity field determination also contributes to the efficient computation of the Molodensky series terms. In this study, the spectral 1D FFT (Haagmans et al., 1993) method on the sphere is combined with the remove-restore procedure. The height anomaly computation is performed according to the eq.:

$$\Delta g_P = g_P - \gamma_Q, \quad \Delta g' = \Delta g_P - \Delta g_P^{GM} - \Delta g_P^{H}, \quad \zeta = \zeta' + \zeta^{GM} + \zeta^{H}, \tag{1}$$

P is the observation point at the Earth's surface and Q is the corresponding point on the telluroid. The ζ_{GM} term gives the contribution of the geopotential model (EGM96), while the ζ_H term gives the contribution of the topography. The term ζ' gives the contribution of the reduced free-air gravity anomalies with the effects of the geopotential model and the topography removed.

3. Molodensky's series

In Molodensky's theory, the height anomaly can be expressed as a series (Brovar, 1964; Moritz, 1980; Li et al., 1995) of the form:

$$\zeta' = \zeta_0 + \zeta_1 + \zeta_2 + \zeta_3 + \square, \tag{2}$$

where

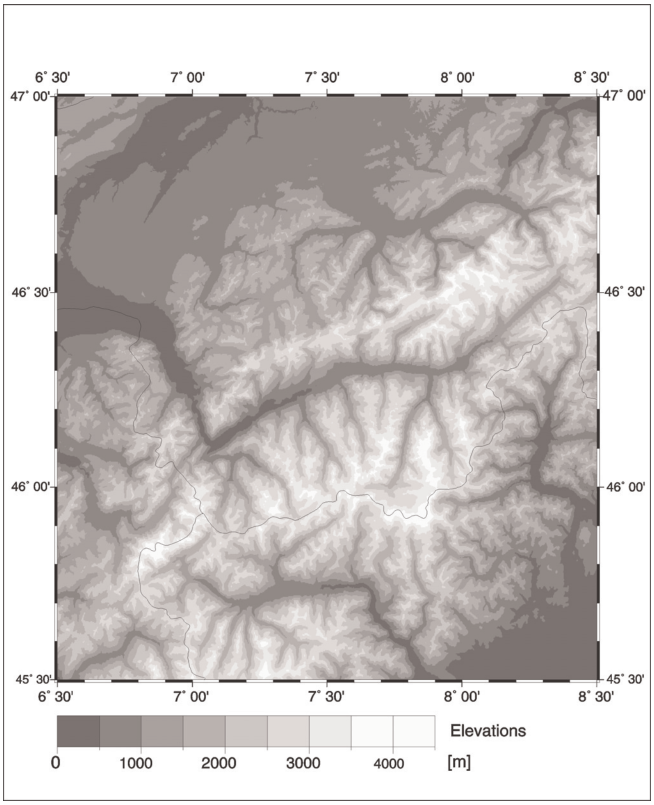

Fig. 1 - The 7.5"×7.5" DTM for the test area (Alps).

$$\zeta_n = \frac{R}{4\pi\gamma}\iint_E G_n S(\psi)d\sigma, n = 0,1,$$

$$\zeta_n = \frac{R}{4\pi\gamma}\iint_E G_n S(\psi)d\sigma - \frac{R^2}{4\pi\gamma}\iint_E \frac{(h-h_P)^2}{l_0^3} G_{n-2}d\sigma, n = 2,3. \tag{3}$$

G_0=$\Delta g'$ are the gravity anomalies on the telluroid. The next three terms take the form:

$$G_1 = \frac{R^2}{2\pi}\iint_E \frac{h-h_P}{l_0^3}G_0 d\sigma, \quad G_2 = \frac{R^2}{2\pi}\iint_E \frac{h-h_P}{l_0^3}G_1 d\sigma + G_0 \tan^2\beta, \tag{4}$$

$$G_3 = \frac{R^2}{2\pi}\iint_E \frac{h-h_P}{l_0^3}G_2 d\sigma + G_1 \tan^2\beta - \frac{3R^2}{4\pi}\iint_E \frac{(h-h_P)^3}{l_0^5}G_0 d\sigma, \tag{5}$$

where l_0=$2R \sin(\psi/2)$, ψ is the spherical distance between the running point and the computation point, and β is the total terrain inclination angle. More details about this formulation and its practical implementation can be found in Li et al. (1995). All the integrals in the above formulas

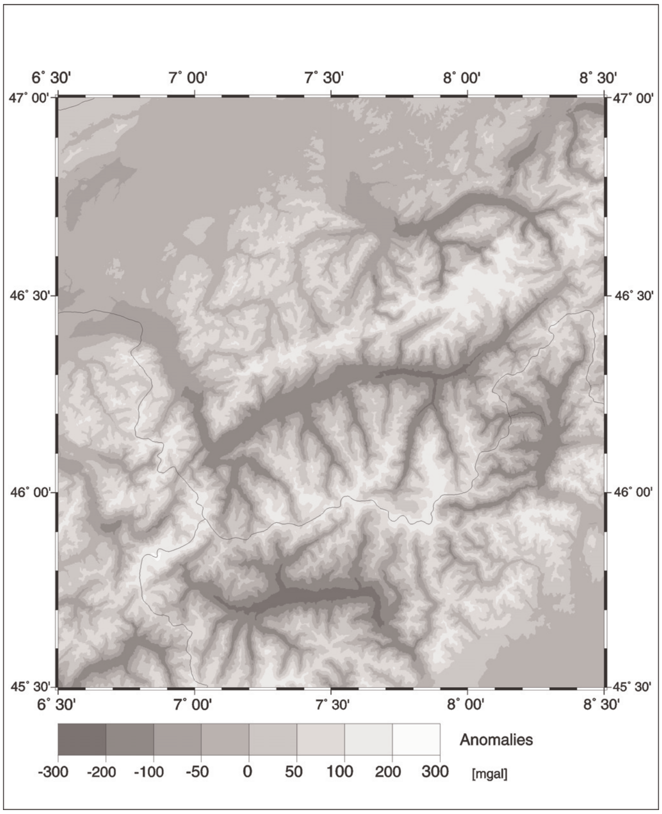

Fig. 2 – Free-air gravity anomalies in the Alps test area (after the subtraction of EGM96).

are evaluated by the efficient 1D FFT technique.

4. Numerical results and discussion

A mountainous 1.5°×2.5° area in the European Alps is chosen for the numerical tests. This area in the European Alps (45.5° N - 47.0° N, 6.5° E - 8.5° E) includes the Mont Blanc massif, being the highest peak in the entire European Alps.

The following topographic height and gravity grids were used: 7.5"×7.5", 15"×15", 30"×45", 60"×90", 2'×3', 5'×5'. The statistics of these Digital Terrain Models (DTMs) as well as the statistics of the inclinations of each terrain model are shown in Table 1. Fig. 1 shows the 7.5"×7.5" DTM for the test area. The numerical examples include the traditionally unreduced observations as well as various terrain-reduced data using the simple terrain correction (t.c.), the complete topographic (Bouguer) reduction (TOPO), the topographic-isostatic reduction (TOPO/ISO) and the RTM reduction. Figs. 2 and 3 show the free-air and RTM30-reduced gravity anomalies after the subtraction of the global model EGM96. Table 2 gives the statistics of the ζ_0, ζ_1, ζ_2, ζ_3 terms for the 30"×45" (1 km x 1 km) height and gravity grid. The gravity data are used in an unreduced form (only the removal of the EGM96 geopotential model is made) as well as

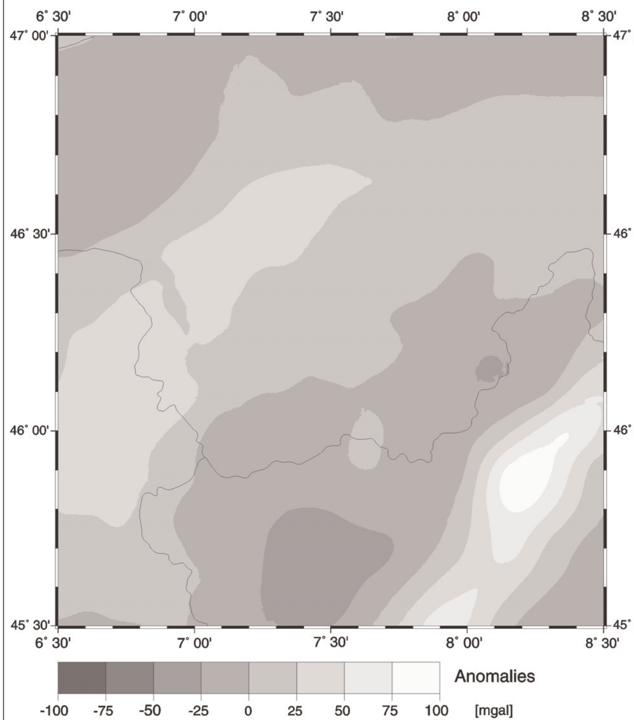

Fig. 3 - RTM30-reduced gravity anomalies in the Alps test area (after the subtraction of EGM96).

after applying different kinds of terrain reductions described above.

The results of Table 2 show that the expected smoothing of the gravity data due to the different reductions results in a corresponding reduction of the magnitude of the ζ_n -terms, when comparing with the corresponding results of the unreduced data. The smoothing of the gravity data also contributes to a faster convergence of the Molodensky series. This is most significant for the RTM-technique either using a 15' or 30' reference topography grid (RTM15 and RTM30 resp., see Table 2). In Table 3, the statistics of G_n as well as the ζ_n terms are summarized for the

Table 1 - Statistics of the DTMs for the Alps area.

Grid Size	No	Elevations				Terrain Inclinations	
		Mean [m]	σ [m]	Min [m]	Max [m]	σ [°]	Max [°]
7.5"×7.5"	691 200	1541.7	834.1	163.0	4767.0	11.4	69.6
15"×15"	172 800	1541.7	832.8	164.0	4701.3	10.3	63.0
30"×45"	28 800	1541.7	826.9	166.3	4548.9	8.2	51.2
60"×90"	7 200	1541.7	813.4	168.6	4216.5	6.1	34.6
2'×3'	1 800	1541.7	785.9	171.5	3791.2	3.9	25.1
5'×5'	432	1541.7	738.2	197.7	3364.7	2.0	10.6

Table 2 - Statistics of the ζ_n terms for the Alps test area (30"×45").

Quant.	Top. Red.	Mean [m]	σ [m]	Rms [m]	Min [m]	Max [m]
ζ_0	-	-0.391	0.463	0.606	-2.179	0.677
ζ_1	-	1.007	0.255	1.038	0.446	1.507
ζ_2	-	-0.016	0.012	0.020	-0.064	0.019
ζ_3	-	-0.054	0.016	0.056	-0.108	-0.022
ζ_0	TOPO	-12.874	2.556	13.126	-17.357	-6.786
ζ_1	TOPO	-0.033	0.099	0.104	-0.344	0.412
ζ_2	TOPO	-0.029	0.019	0.035	-0.131	0.051
ζ_3	TOPO	0.000	0.002	0.002	-0.024	0.020
ζ_0	TOPO/ISO	-4.856	1.389	5.050	-7.325	-2.376
ζ_1	TOPO/ISO	-0.027	0.044	0.052	-0.199	0.173
ζ_2	TOPO/ISO	-0.016	0.010	0.019	-0.071	0.017
ζ_3	TOPO/ISO	0.000	0.001	0.001	-0.008	0.004
ζ_0	RTM15	0.747	0.432	0.863	-0.304	1.603
ζ_1	RTM15	0.066	0.018	0.069	-0.021	0.147
ζ_2	RTM15	-0.001	0.004	0.004	-0.022	0.018
ζ_3	RTM15	-0.001	0.001	0.001	-0.008	0.004
ζ_0	RTM30	0.550	0.331	0.642	-0.277	1.452
ζ_1	RTM30	-0.002	0.012	0.012	-0.062	0.037
ζ_2	RTM30	-0.001	0.002	0.003	-0.011	0.012
ζ_3	RTM30	0.000	0.000	0.000	-0.002	0.002

30"×45" and the 7.5"×7.5" data grids associated with the RTM30-reduction technique. Studying simultaneously the results of Tables 2 and 3, it is apparent that for the 30"×45" data grids mainly the ζ_0 (G_0) terms are significant in height anomaly predictions. The contribution of the ζ_1 (G_1) terms reaches the level of about 10 cm. The ζ_2 (G_2) terms contribute up to 1 cm to the height anomalies, while the contribution of the ζ_3 (G_3) terms is almost negligible. Comparing these results with those obtained for the unreduced data, one can observe that for the unreduced data not only the first two terms but also the ζ_2 (G_2) and ζ_3 (G_3) terms contribute up to 11 cm to the height anomaly prediction.

In order to further investigate the effects of the grid spacing on the height anomaly predictions, numerical evaluations are also done for the dense 7.5"×7.5" grid, being available in the Alps test area. Again, the statistics are given in Table 3 only for the RTM-corrected gravity data. The statistics of the results are comparable to the corresponding results based on the 30"×45" grid. As before, only the first two terms mainly contribute to the height anomaly prediction, while the contribution of the ζ_2 (G_2) and ζ_3 (G_3) terms is considerably smaller (reaches the level of 3 cm).

From all these results one can conclude that the smoothing of the gravity data by the different gravity reductions and mainly by the RTM procedure leads to a considerable reduction of the magnitude of the ζ_n (G_n) terms. It is worth mentioning that in Table 3 some of the

Table 3 - Statistics of the G_n and ζ_n terms for the Alps test area using RTM30-reduced data.

	30''×45'' No: 28,800					7.5''×7.5'' No: 691,000				
	Mean [mGal]	σ [mGal]	Rms [mGal]	Min [mGal]	Max [mGal]	Mean [mGal]	σ [mGal]	Rms [mGal]	Min [mGal]	Max [mGal]
G_0	6.72	21.28	22.32	-46.94	94.29	6.72	21.29	22.32	-47.07	94.40
G_1	-0.01	5.59	5.59	-47.65	35.62	-0.01	9.67	9.67	-130.36	114.82
G_2	0.00	1.00	1.00	-8.27	12.17	0.09	2.98	2.98	-49.51	95.68
G_3	0.00	0.48	0.48	-7.54	8.08	0.00	2.27	2.27	-190.15	273.49
	Mean [m]	σ [m]	Rms [m]	Min [m]	Max [m]	Mean [m]	σ [m]	Rms [m]	Min [m]	Max [m]
ζ_0	0.550	0.331	0.642	-0.277	1.452	0.550	0.332	0.643	-0.280	1.457
ζ_1	-0.002	0.012	0.012	-0.062	0.037	-0.002	0.013	0.013	-0.080	0.044
ζ_2	-0.001	0.002	0.003	-0.011	0.012	0.006	0.007	0.009	-0.021	0.034
ζ_3	0.000	0.000	0.000	-0.002	0.002	0.000	0.000	0.000	-0.011	0.014

minimum/maximum values of the G_n terms ($n \geq 1$) are unexpectedly large. This implies that, in very rough mountainous areas, the grid spacing should not be too small in order to avoid the instability problem related to the Molodensky series. It is known that the Molodensky series converges only when the terrain inclination angle is less than 45° (Li et al., 1995), which cannot be guaranteed if the grid spacing is too small in rugged terrain. A grid size of about 1 km appears to be a reasonable value when using terrain-reduced data.

In another numerical test the computed height anomalies were compared with corresponding values derived by GPS/leveling. In the case that only orthometric heights are available, a trans-

Table 4 - Statistics of the differences between GPS/leveling and the predicted $\zeta(n)$.

Quantity	Top. Red.	No	Bias fit (1 par.) Rms	Min	Max	Bias and tilt fit (3 par.) Rms	Min	Max
$\zeta(0)$	-	25	0.471	-0.813	0.673	0.132	-0.315	0.273
$\zeta(1)$	-	25	0.201	-0.431	0.292	0.078	-0.193	0.193
$\zeta(2)$	-	25	0.211	-0.440	0.312	0.079	-0.192	0.203
$\zeta(3)$	-	25	0.228	-0.465	0.335	0.081	-0.196	0.210
$\zeta(0)$	t.c.	25	0.130	-0.292	0.286	0.085	-0.154	0.178
$\zeta(0)$	RTM30	25	0.158	-0.245	0.319	0.092	-0.204	0.170
$\zeta(1)$	RTM30	25	0.164	-0.245	0.337	0.094	-0.198	0.185
$\zeta(2)$	RTM30	25	0.164	-0.246	0.339	0.094	-0.197	0.186
$\zeta(3)$	RTM30	25	0.164	-0.246	0.339	0.094	-0.197	0.186
EGG97	RTM30	25	0.064	-0.090	0.208	0.056	-0.096	0.164

formation to normal heights was done first using a correction in the form $\Delta g_B H/\gamma$, where Δg_B is the Bouguer anomaly at the computation point, H is the orthometric height and γ is the normal gravity. The statistical results of the such derived differences are tabulated in Table 4 after doing a bias fit (1 parameter) as well as a bias and tilt fit (3 parameters). The fitting is made in order to absorb long wavelength systematic errors (Li et al., 1995). In Table 4, the term $\zeta(n)$ is used for the complete height anomaly including the Molodensky terms up to the order n. For the unreduced data the differences become smaller when the ζ_1 terms are included, and they remain almost the same when also the higher order terms are taken into account. For the RTM-corrected data, both the RMS difference and the difference range (max - min) are about the same when ζ_1, ζ_2 or ζ_3 are included. In general, considering also results in other test areas, the RMS values become smaller when the ζ_n terms ($n \geq 1$) are included.

Two more comparisons are given in Table 4. The first one is between the GPS/leveling derived height anomalies and the gravimetric height anomalies computed from gravity data corrected by the simple terrain correction (i.e. Faye anomalies that are used in many studies as a first order approximation to the Molodensky series). The RMS difference is slightly smaller than the value found for the unreduced or RTM-corrected gravity data. Another comparison is made between GPS/leveling and the height anomalies of the new European solution EGG97 (Denker and Torge, 1997) based on RTM-corrected gravity data. This comparison gives the best results (smallest values) both for the RMS difference and the difference range (max – min), showing the benefit of using data in a much larger area.

5. Conclusions

Terrain reductions should be applied in the process of gravity data gridding. The gravity reductions can remove the high frequency information present in gravity observations. These smoothed gravity data yield significantly reduced Molodensky series terms ζ_n ($n \geq 1$) and an improved series convergence. From the reductions used in this study, the RTM technique gave the smoothest results and consequently the smallest ζ_n (G_n) terms. From a practical point of view, only the ζ_0 and ζ_1 are needed in the Alps test area. The maximum effect of the ζ_n terms ($n \geq 1$) is smaller than 10 cm in the mountainous Alps and less than 1 cm in low mountain ranges. Thus the magnitude of the Molodensky terms is not very critical in connection with terrain-reduced gravity field data. However, these terms should be included in future precise height anomaly computations such as the European Gravimetric (Quasi) Geoid. The dependency of the Molodensky series terms on the grid spacing also reduces considerably for terrain-reduced gravity field data. The high resolution DTMs (e.g. 100 m grids) may cause numerical instabilities in the Molodensky series in rough mountainous areas, especially in the computation of the G_n terms. However, this does not affect the height anomaly prediction very much, but further investigation is necessary in this direction. The agreement of the height anomalies with corresponding values from GPS/leveling is slightly improved when considering the Molodensky series terms ζ_n ($n \geq 1$).

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