BOLLETTINO DI GEOFISICA TEORICA ED APPLICATA VOL. 40, N. 3-4, pp. 589-595; SEP.-DEC. 1999

RTC and density variations in the estimation of local geoids

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(Received August 4, 1998; accepted October 5, 1999)

Abstract. In the frame of a project aiming to the revision of the estimation procedures for local geoids, we have been studying the algorithms for the RTC systematically. In this contribution, the work carried out on a small DTM test case (5 Km × 5 Km) is presented; the work focuses especially on two topics: verifying the effect of the surface density variations (continuous as well as discontinuous) on the computation of the RTC term and evaluating the errors introduced in the RTC computation by alternative approximations applied to the density variations modeling.

1. Introduction

In the remove-restore methods for geoid computation, the Residual Terrain Correction is of fundamental importance. This step can be thought of as the difference between two separate terrain correction computations: the first one is performed on the real DTM, the second, on an averaged DTM. So, despite the fact that our interest is centered on RTC, in the following we will always refer to TC computation.

In the frame of a project aiming to revise the estimation procedures for local geoids, we are systematically studying the algorithms for the RTC computation; our research focuses on two aspects:

- verifying the effects of the surface density variation (continuous as well as discontinuous) on the computation of the term RTC;
- verifying the potentiality (precise results and computational time) of a new algorithm, recently developed to speed up the computation of RTC with a very fine DTM grid.

In this note we address our attention to the first research topic; for the other one we refer to

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Sansò et al. (1998). We only wish to recall that our new formulas are based on the analytical splitting of TC integral into several terms: for each term a suitable approximating formula, which minimizes estimate errors and computation time is found.

So, at present the computation is performed along two alternative approaches:
- the DTM is assumed to be representative of a set of prisms whose gravitational attraction can be estimated by closed formulas at each computational point (Forsberg, 1994);
- the exact formulas are simplified to transform them into a sum of convolution integrals to be estimated by FFT (Sideris, 1994).

The first approach only introduces the approximation of considering the real topography composed of a set of discontinuous shapes, while it is a continuous surface almost everywhere else; on the other hand, it is numerically slow when fine DTM grids are used in the computation of TC for a large amount of gravity points (as happens in geoid computation); the second approach is very fast, but it is based on approximations on slopes on the terrain whose effects can be significant in rugged topography, such as in mountainous areas.

2. The problem

As is well known in planar approximation, the following formula for the gravitational attraction for topographic masses, holds

$$g(\mathbf{x},h_x) = G\int\int_{R2} d\xi \int_0^{h\xi} d\zeta \left[-\frac{\partial}{\partial h_x} \frac{\rho(\xi,\zeta)}{\left[|\xi-\mathbf{x}|^2+(h_x-\zeta)^2\right]^{1/2}} \right] \tag{1}$$

where G is the universal gravitational constant; $\mathbf{x}$ is the planimetric position of the computational point while h_x is its height; ξ and ζ are the coordinates of the integration point; ρ is the terrain density.

The first approximation on density, applied in any efficient numerical computation of the TC, consists in assuming that the density is not a function of the depth: so the Eq. (1) becomes

$$g(\mathbf{x},h_x) = G\int\int_{R2} d\xi\rho(\xi) \left[\frac{1}{\left[|\xi-\mathbf{x}|^2+\left(h_x-h_\xi\right)^2\right]^{1/2}} \frac{1}{\left[|\xi-\mathbf{x}|^2+h_x^2\right]^{1/2}} \right] = I\left[\rho(\xi)\right] \tag{2}$$

Note that a straightforward representation for the density in Eq. (2) is $\rho(\xi)=\rho_C+\delta\rho_C$ where ρ_C is a constant value; $\delta\rho_C(\xi)$ takes into account the density variation with respect to ρ_C. This representation splits the right hand side of Eq. (2) into the sum of two integrals:

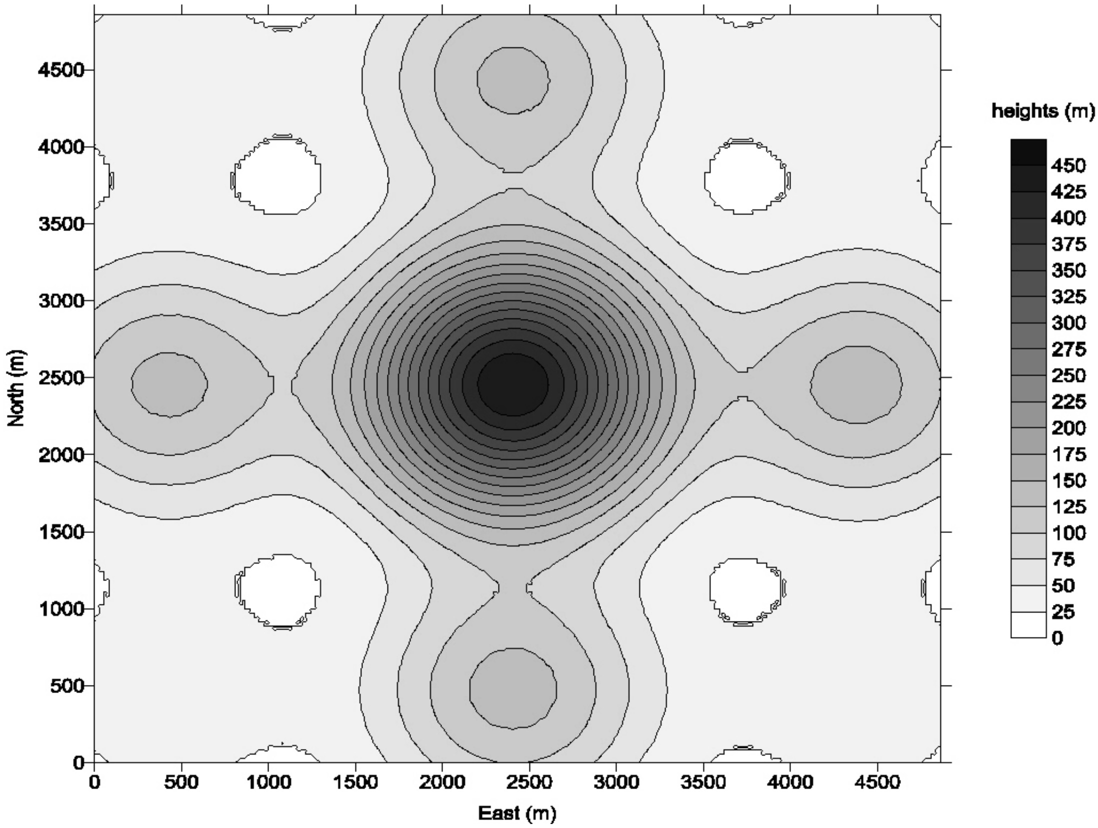

Fig. 1 - Simulated DTM.

$$g(\mathbf{x},h_x)=I[\rho_C]+I[\delta\rho_C(\xi)]$$

Usually, the density is assumed to be a suitable constant over the whole computational domain (typically 2.67 gr/cm^3) and the term $I[\delta\rho_C]$ is neglected (i. e. $\delta\rho_C(\xi)$ is assumed to be zero everywhere); so the TC is computed following one of the approaches (prisms integration, FFT, new formulas) outlined in the introduction. It is worthwhile to note that the approximation $\delta\rho_C = 0$ is not necessary for the numerical estimate of the TC by prism integration or by FFT, even if adopted by all the available software. On the contrary, the analytical splitting of the TC integral (2), on which our method is based, requires the density not to vary in the integration domain; otherwise, other corrective terms must be added taking into account the existence of density variations.

At present, information on the surface density is available for some areas (see, for instance, Baiocchi et al. (1998) for the Italian territory): at a detailed level, this information allows us to know not only, the mean surface density, but also its variation as a function of planimetric coordinates; so it is useful to verify the existence of an optimal choice for the value of ρ_C, namely the one that minimizes the omission error (i. e. the error due to the hypothesis $\delta\rho_C = 0$); moreover, it is useful to evaluate the sensitivity of the terrain correction formula to the density variation, in order to investigate on the opportunity of implementing some corrective terms for

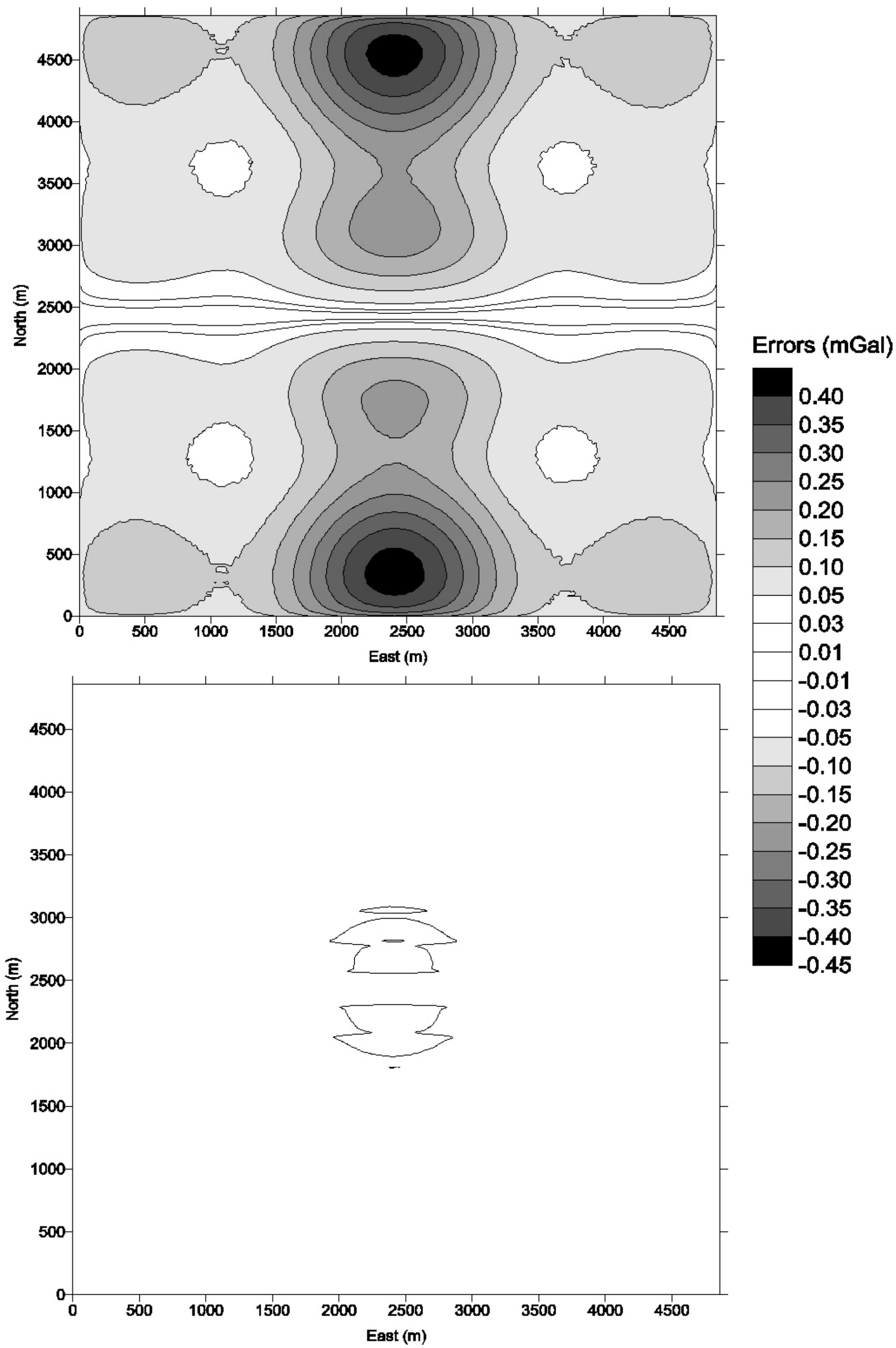

Fig. 2 - Linear density variation, A.2 (upper) and A.3 (bottom) approximation errors.

Table 1 - Comparison fo the resultes dotained by different approximation: A.1, A.2 and A.3.

Data Type	E meters for DTM; mGal for others	σ meters for DTM; mGal for others	min meters for DTM; mGal for others	Max meters for DTM; mGal for others
Simulated DTM	86	78	22	449
$\rho_L(\xi)$ Exact TC	8.287	6.949	1.531	38.674
A.2 Errors	< 0.001	0.143	-0.427	0.425
A.3 Errors	< 0.001	0.005	-0.025	0.024
$\rho_S(\xi)$ Exact TC	8.284	6.942	1.584	38.791
A.2 Errors	-0.001	0.330	-0.945	0.984
A.3 Errors	-0.001	0.120	-1.174	1.191
$\rho_{200}(\xi)$ Exact TC	8.284	6.942	1.584	38.791
A.2 Errors	0.003	0.326	-0.939	0.978
A.3 Errors	< 0.001	0.090	-0.830	0.978
$\rho_{800}(\xi)$ Exact TC	8.285	6.942	1.584	38.633
A.2 Errors	0.002	0.291	-0.845	0.885
A.3 Errors	< 0.001	0.037	-0.302	0.279

the omission error also in our new formulas.

This is precisely our problem. Surprisingly, it will turn out that although the use of a blunt constant density function is in many instances not sufficient, there is the possibility of using a corrective multiplying factor which is effective and requires no more effort than the one needed by the usual software.

3. Numerical simulation and results

To investigate the problem we have simulated a DTM on a $5x5Km^2$ grid, 25-meter spaced both in North-South and East-West directions (Fig. 1). On the same grid we have also simulated several different density models presenting North-South variations:

1. the first one, $\rho_L(\xi)$, is a linear trend from North to South, with a minimum value of 2.6 gr/cm^3 in the northern area and a maximum of 2.8 gr/cm^3 in the southern area;
2. the second, $\rho_S(\xi)$, is a stepwise function with a constant value of 2.6 gr/cm^3 in the upper half of the grid and 2.8 gr/cm^3 in the lower half;
3. the other two models $\rho_{200}(\xi)$ and $\rho_{800}(\xi)$ considered have constant density values in the North and Southern areas (as $\rho_S(\xi)$) and a central transition stripe, respectively 200m and 800m wide, in which the density follows a linear trend as for $\rho_L(\xi)$.

A specific software, using the prisms method, has been implemented to calculate the terrain correction associating each prism's height and density. For each density model, for each node ($\mathbf{x}_{i,j}$) of the grid, the following TC estimates have been carried out:

- A1. the "exact" TC due to the simulated DTM, namely the one obtained using, each prism's simulated density: $g(\mathbf{x}_{i,j}, h_x)=I\,[\rho(\xi_{k,l})]$ in Eq. (2);

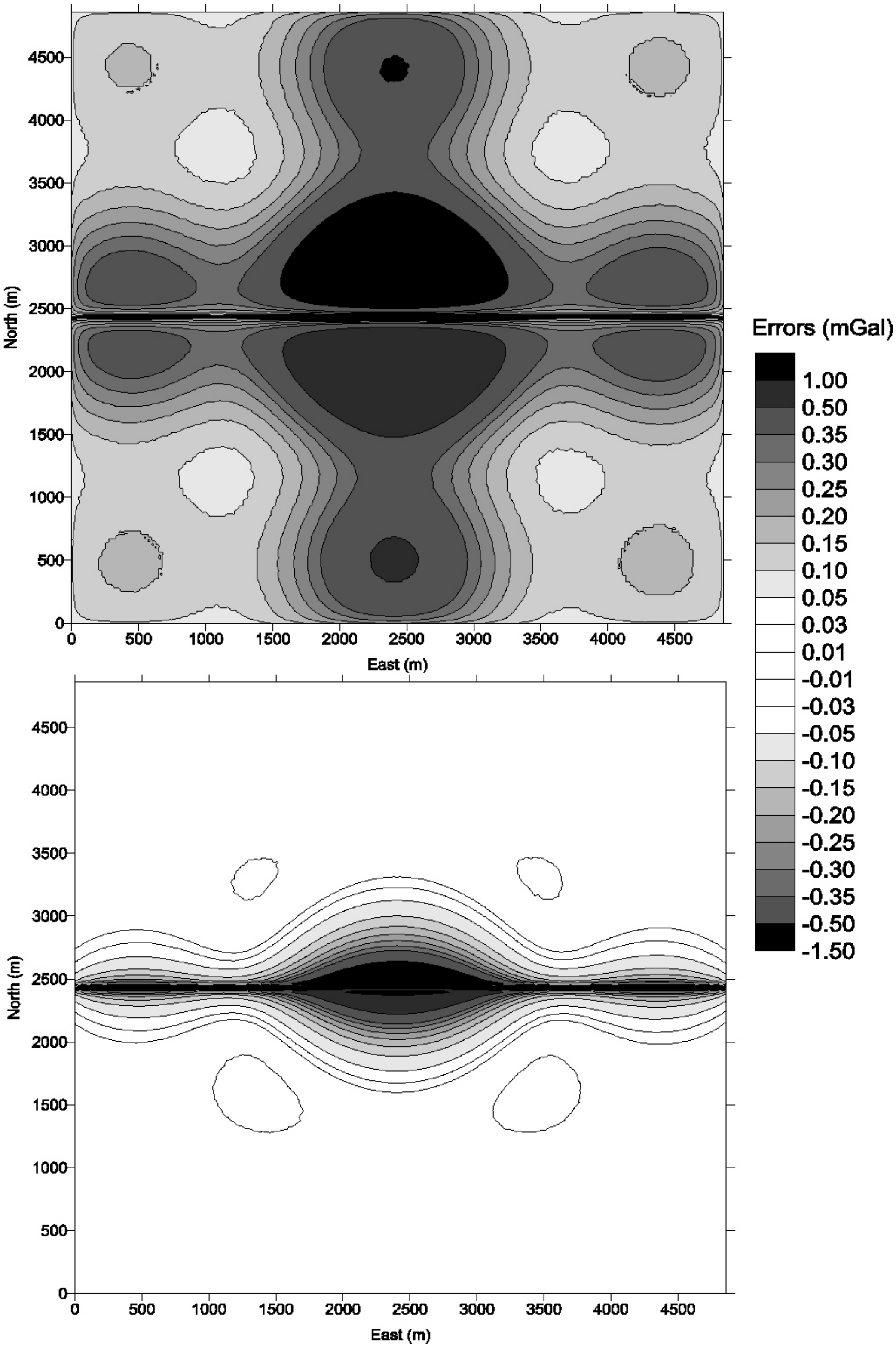

Fig. 3 - Stepwise density variation, A.2 (upper) and A.3 (bottom) approximation errors.

- A2. the approximated TC, obtained using each prism's the mean density on the computation domain: $g(\mathbf{x}_{i,j}, h_x)=I\ [\rho_C)]$ where $\rho_C = \rho$;
- A3. the approximated TC, obtained associating to each prism whose effect has to be computed, the density corresponding to the computation node and considered as constant through all the domain: $g(\mathbf{x}_{i,j}, h_x)=I\ [\rho_C]$ where $\rho_C=\rho(\mathbf{x}_{i,j})$;

Note that the approximation A.2 is the classical one adopted in the TC computation; the approximation A.3 represents an alternative approach not requiring substantial modifications to existing programs: indeed, the TC integrals involved in the two approaches, are related simply by

$$I\left[\rho\left(\mathbf{x}_{ij}\right)\right] = \frac{\rho\left(\mathbf{x}_{ij}\right)}{\bar{\rho}} I\left[\bar{\rho}\right]$$

due to our hypothesis that $\rho(\xi_{i,j})$ is constant over the domain.

For each density model, we have compared the "exact" (A.1) results with those obtained by the two different approximations A.2 and A.3. (Figs. 2-3, Table 1).

The results obtained analyzing the $\rho_L(\xi)$ density model, clearly show that the approximation $\rho(\xi)=\rho$ brings significant omission errors everywhere, except in the middle of the grid, where the actual density coincides with the mean density. Otherwise, the approximation $\rho(\xi)=\rho(\mathbf{x}_{i,j})$ improves the results everywhere.

In the case of the discontinuous density model, the approximation A.3 as compared with A.2, greatly improves the results in a mean square sense, as can be seen from the statistics; although it gives rise to greater omission errors in the proximity of the discontinuity line. However, as in the $\rho_{200}(\xi)$ and $\rho_{800}(\xi)$ models, the errors furnished by solution A.3 decrease everywhere with respect to those provided by A.2, when the transition zone "relaxes".

4. Conclusions

In the real world, the density transitions are often discontinuous; however, our present knowledge of surface density is not so granular, and we can try to define only its large or, at best, medium scale variations. So, by adopting approximation A.3, analyzed in this work, seems to significantly improve the TC results in the majority of real cases, without increasing the computation much. Anyway, when abrupt density discontinuities occur and accurate information on them are available, a first-order modeling of their gravitational effects may be useful.

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