The characteristics of a nonlinear elastic cylindrical wave

A. AKINOLA

Mathematics Department Obafemi Awolowo University, Nigeria

(Received October 10, 1997; accepted June 22, 1998)

Abstract. We study wave propagation in an elastic layer under finite deformations. The variation of the propagating wave speeds with respect to the parameter of finite deformation is highlighted. We give the characteristic curves for the linear cylindrical wave and the nonlinear coupled wave: the diagrams giving their shape are constructed, depending on the mode of initial excitation.

1. Introduction

Cylindrical objects are essential materials (components) in the oil prospection industry and other areas of applied geophysics. They are often used as research media (Carcione and Seriani, 1994 and Kaduchak et al., 1996); and so it is pertinent, for instance, to have a qualitative knowledge of the waves propagating in them. Furthermore, these and other related media of interest, like the earths crust, wich is the location of mineral deposits, including hydrocarbon reservoirs, are frequently modelled as layered/anisotropic media (Carcione and Cavallini, 1993; Honarvar and Sinclar, 1996; Kohler et al., 1996). What is more, just as the porous media modelling (Carcione and Quiroga-Goode, 1996) of the oil reservoir and sedimentary environment is gaining impetus these days so also is the need to examine finite deformation (or nonlinear) effects (Lyakhovsky et al., 1997 and Barclay, 1998) on physical phenomena in them and their properties. Although, a good number of these media under consideration are not only nonlinear in mechanical behaviour but also inelastic or time dependent (Carcione and Quiroga-Goode, 1996; Crampin and Zatsepin, 1997; Zatsepin and Crampin, 1997; Carcione, 1997), nevertheless elastic modelling (Kaduchak et al., 1996; Kohler et al., 1996; Lyakhovsky et al., 1997) is the first window through which analytical information is exploited. This is so, in part, to avoid a totally complicated, if not clumsy, situation from the onset.

Justifiably, in almost all aspects of geophysical endeavour, and elsewhere, the behaviour of the *P-waves* and *S-waves*, in one form or an other, is of sustained interest (Vavryčuk and

---

Corresponding author: A. Akinola; Mathematics Department, Obafemi Awolowo University, ILE-IFE, Nigeria

© *1998 Osservatorio Geofisico Sperimentale*

Yomogida, 1996; Carcione, 1997; Castro et al., 1997; Igel et al., 1997 and Barclay, 1998). They are basic waves and are continually used to investigate phenomena and deduce analogies (Carcione and Cavallini, 1995). Hence the need to investigate their behaviour when finite deformation is taken into consideration. This, amongst other things, may influence the quality of signals between sources and targets in geophysical experiments, such as in mineral exploration where petrophysical property changes that induce acoustic property changes in hydrocarbon reservoirs are detected by reflection seismic experiments.

In this work the effect of finite deformations on elastic waves is studied in a layered medium.

The main difficulty with problems in finite deformations is getting a workable energy function or a constitutive law (Green and Adkins, 1960 and Lurie, 1980). But once this is obtained, by whatever means, the difficulty associated with nonlinearity of the consequent boundary value problem can then be tackled with available mathematical methods. From both the practical and theoretical points of view the consideration of finite deformations is of interest, since amongst other things, it enables some effects, often supressed through the small (or infinitesimal) deformation approach, to be detected. Knowledge of some of these effects at times can be of immense value, not only in the fields of seismology, and oil and other related mineral exploiting industries, but also in water resources and others. An important point here is the fact that there exists a nonlinear elastic wave, even in an infinite homogeneous isotropic medium, to the extent that the longitudinal and shear waves are coupled (Ibitoye and Akinola, 1993).

Here, we invoke the mathematical theory of characteristics, wich indicates that under appropriate conditions an hyperbolic partial differential equation can be decomposed into ordinary differential equations along curves known as *characteristics*. Based on this, without having directly solved any boundary value problem as it were (Courant and Hilbert, 1962; Achenbach, 1973 and Whitham, 1974), we attempt to give the expression for the wave speeds, and the characteristics for the linear longitudinal wave and the nonlinear coupled wave, the later being an effect of finite deformation.

2. Propagating waves for infinitesimal deformation

We recall the classical theory for infinitesimal (small) deformation. The motion of particles in an homogeneous isotropic elastic medium is given by the classical Lame's equation:

$$(\lambda+2\mu)\nabla(\nabla\cdot\vec{u})-\mu(\nabla\times\nabla\times\vec{u})+\rho_o\vec{f}=\rho_o\frac{\partial^2}{\partial t^2}\vec{u}. \tag{1}$$

By Helmholtz's theorem on the decompostion of a vector into a scalar potential and a vector potential for the displacement vector $\vec{u}$ and for the body force vector $\vec{f}$

$$\vec{u}=\nabla\varphi+\nabla\times\vec{\psi};\;\; \vec{f}=\nabla\Phi+\nabla\times\vec{\Psi}, \tag{2}$$

and since $\nabla\times(\nabla\varphi)$=0, $\nabla\cdot(\nabla\times\vec{\psi})=0$, for any scalar $\varphi$and any vector $\vec{\psi}$, we obtain from Eq. (1), provided $\nabla \cdot \vec{\psi} = 0$, the two uncoupled motion equations

$$c_L^2 \nabla^2 \varphi + \frac{\Phi}{\rho_o} = \frac{\partial^2 \varphi}{\partial t^2}, \tag{3}$$

$$c_S^2 \nabla^2 \vec{\psi} + \frac{\vec{\Psi}}{\rho_o} = \frac{\partial^2 \vec{\Psi}}{\partial t^2}, \tag{4}$$

with the longitudinal wave (or dilatational wave or p-wave) speed and the shear wave (or transverse wave or s-wave) speed respectively

$$c_L \equiv \left(\frac{\lambda + 2\mu}{\rho_o} \right)^{1/2}, \qquad c_s \equiv \left(\frac{\mu}{\rho_o} \right)^{1/2}, \tag{5}$$

λ are μ the Lame's elastic constants and ρ_o is the material density.

We see that the scalar potential φ, such that $\vec{u} = \nabla\varphi$, is associated with the longitudinal wave, while the vector potential $\vec{\psi}$, such that $\vec{u} = \nabla \times \vec{\psi}$, is associated with the shear wave. We also note that Eq. (3) is a scalar relation, while Eq. (4) is vectorial, from which we can obtain the scalar relations equivalent to Eq. (3). In fact, consider $\vec{u} = \nabla \times \vec{\psi}$ and note that $\nabla \times \nabla \times \vec{\psi} = \nabla (\nabla \cdot \vec{\psi}) - \nabla^2 \vec{\psi}$. Then noting that $(\nabla^2 \vec{\psi})_i = (\nabla \times \nabla \times \vec{\psi})_i = \varepsilon_{ijk}\partial_j (\nabla \times \vec{\psi})_k = \varepsilon_{ijk}\partial_j u_k = \varepsilon_{ijk}\partial_j (\varepsilon_{kmn}\partial_m \psi_n)$ we obtain

$$(\nabla^2 \vec{\psi})_i = \varepsilon_{ijk}\varepsilon_{kmn}\partial_j \partial_m \psi_n; \;\; i,j,k,l,m,n = 1,2,3, \tag{6}$$

where ε_{ijk} is the Levi-Civita anti-symmetric rotation tensor, and ∂_i is the partial derivative in the pertinent argument along the direction of the i-axis.

Now, in rectangular coordinates (x_1, x_2, x_3), in the case of a plane wave, $u\,(x, t)$ Eq. (6) reduces to $\nabla^2 \vec{\psi} = \partial_1^{\,2} \psi_2 \vec{j} + \partial_1^{\,2} \psi_3 \vec{k}$. Substituting this in Eq. (4), we obtain the usual two similar wave equations, the horizontally polarized shear wave or SH-wave and the vertically polarized shear wave or SV-wave:

$$c_s^2 \partial_1^2 \psi_2 = \frac{\partial^2 \psi_2}{\partial t^2}; \qquad c_s^2 \partial_1^2 \psi_3 = \frac{\partial^2 \psi_3}{\partial t^2}. \tag{7}$$

Likewise, in cylindrical coordinates (r, θ, z), $\vec{\psi} = (\psi_r, \psi_\theta, \psi_z)$ and for plane waves $\vec{u} = \vec{u}(r, t)$ Eq. (6) again gives a similar pair of wave equations as in Eq. (7), for the rotary shear motion and axial shear motion respectively:

$$c_s\left(\frac{\partial^2 \psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_r}{\partial r} - \frac{\psi_r}{r^2}\right) = \frac{\partial^2 \psi_r}{\partial t^2}; \qquad c_s\left(\frac{\partial^2 \psi_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r^2}\right) = \frac{\partial^2 \psi_\theta}{\partial t^2}. \tag{8}$$

Thus, under the small deformation assumption, the two propagating waves, the P-wave and S-wave (SH or SV), are linear and uncoupled.

3. Wave equations for finite deformation

3.1. Problem setting

Let Ω be a subset of a 3-dimensional Euclidean space E^3 (i.e. $\Omega \subset E^3$). Consider Ω as an infinite composite medium consisting of concentric periodic cylinders Ω_m ; $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega$, where m is any natural number.

We seek the plane finite deformation of Ω from an initial configuration Ω_0 onto a current configuration Ω by the action, say, of an excitation on the interior surface (i.e. of the innermost cylinder) radius r_0 , in the form (Sujunshkaliev, 1980)

$$f = f(r,t), \qquad \phi(r,t) = \alpha + \theta(r,t), \tag{9}$$

where r is the radial coordinate, t is the time, α is any initial constant angle and θ is the angular displacement.

The position vectors of every particle in the initial (Ω_o) and the current (Ω) configurations are given respectively by:

$$\vec{r} = r\vec{e}_r + z\vec{k},$$
$$\vec{R} = f(r,t)\vec{e}_R + z\vec{k}, \tag{10}$$

where $\vec{e}_r$, $\vec{e}_\alpha$, $\vec{k}$ are the orthogonal covariant local basis vectors associated with the cylindrical coordinates (r, α, z) in Ω_i, and $\vec{e}_R$, $\vec{e}_\phi$, $\vec{k}$ are the corresponding local basis vectors associated with the cylindrical coordinates (R, ϕ, z) in Ω_c, as a result of the deformation $\vec{r} \to \vec{R}$.

3.2. Geometry of deformation

The gradient tensor of the position vector $\vec{R}$ in Ω ($\vec{R}$) taking in the initial configuration Ω_o ($\vec{r}$) is

$$\overset{\circ}{\nabla} \vec{R} = \vec{r}^{\,i} \frac{\partial}{\partial q_i} \vec{R} = \vec{r}^{\,i} \; \vec{R}_i = f' \, \vec{e}_r \vec{e}_R + f\theta' \, \vec{e}_r \vec{e}_\phi + \frac{f}{r} \vec{e}_\alpha \vec{e}_\phi + \vec{k}\vec{k}. \tag{11}$$

Consider the polar decomposition of the gradient tensor into the symmetric stretch tensor $\tilde{U}$, and the orthogonal (rotation) tensor that is supportive of deformations $\tilde{O}$ such that $\overset{\circ}{\nabla} \vec{R} = \tilde{U} \cdot \tilde{O}$. We note that $\overset{\circ}{\nabla} \vec{R} \cdot \overset{\circ}{\nabla} \vec{R}^{\mathrm{T}} = \tilde{U}^2$, and in view of Eqs. (9), (10) and (11) we obtain

$$\tilde{O} = \frac{1}{D}\Big[\; \frac{1}{r}(rf)' \, (\vec{e}_r \vec{e}_R + \vec{e}_\alpha \vec{e}_\phi + f\theta' \, (\vec{e}_r \vec{e}_\phi - \vec{e}_\alpha \vec{e}_R) + D\vec{k}\vec{k} \;\Big], \tag{12}$$

$$D=\left\{\left[\ \frac{1}{r}(rf)'\right]^2+(f\theta')^2\right\}^{1/2}. \tag{13}$$

Here and elsewhere, for any quantities $\varphi(r, t)$ and $\psi(r, t)$; $\varphi' \equiv \frac{\partial\varphi}{\partial r}$, $\dot{\varphi} \equiv \frac{\partial\varphi}{\partial t}$, and $\varphi\cdot\psi$ and $\varphi\cdot\cdot\psi$ indicate the dot (or inner or scalar) product and double dot product (i.e. two consecutive scalar operations) of the two respectively. $\vec{r}_i=\frac{\partial\vec{r}}{\partial q^i}$, $\vec{r}^i$ is the contravariant vector to $\vec{r}_i$, and $\vec{r}_i\ \vec{r}^j = \delta_{ij}$, where δ_{ij} is the Kronecker delta (unit) tensor, i, j = 1, 2, 3.

The acceleration of the body in the current position, from Eqs. (9) and (10), is

$$\ddot{\vec{R}}=(\ddot{f}-f\dot{\theta}^2)\vec{e}_R+(f\ddot{\theta}+2\dot{f}\dot{\theta})\vec{e}_\phi, \tag{14}$$

where the second terms in the first and second brackets correspond to the centrifugal and the Coriolis accelerations, which may be of some degree of interest in problems in the earth sciences.

3.3. Energy function and motion equation

The deformation Eq. (9) results in a planar problem. Then, by the theory of invariants, this medium will possess 3 invariants corresponding to 2 eigenvectors and a normal vector that characterizes the direction of transversal-isotropy. Therefore, the deformation energy will be a function of 3 invariants of the geometry of deformations, and correspondingly will be characterized by 3 effective moduli (Pobedria, 1984):

$$W=\lambda_2 S_2+\frac{1}{2}\lambda_1 S_1^2+\lambda_o S_o, \tag{15}$$

$$S_o=\vec{c}\cdot\tilde{U}^2\cdot\vec{c},\quad S_1=\tilde{E}\cdot\cdot(\tilde{U}-\tilde{E})\equiv\quad I_1(\tilde{U}-\tilde{E}),\ S_2=I_1(\tilde{U}-\tilde{E})^2; \tag{16}$$

$$\lambda_2=\langle\mu\rangle,\quad \lambda_1=\langle\lambda\rangle+\frac{\langle\frac{\lambda}{(\lambda+2\mu)}\rangle^2}{\langle\frac{1}{(\lambda+2\mu)}\rangle}-\langle\frac{\lambda^2}{(\lambda+2\mu)}\rangle,\quad \lambda_3=\frac{1}{\left\langle\frac{1}{\mu}\right\rangle},\ \lambda_o=\lambda_o(\lambda_2,\lambda_3), \tag{17}$$

where $\vec{c}$ is the unit vector that characterizes the direction of anisotropy. We note that should the medium become non-heterogeneous for whatever reasons, then automatically the energy as in Eq. (15) reduces to its known equivalent for homogeneous (isotropic) body, and same is true for the effective moduli λ_3, λ_2, λ_1, while λ_o vanishes:

$$\lambda_3=\lambda_2=\mu,\quad \lambda_1=\lambda,\quad \lambda_o=0, \tag{18}$$

where λ and μ are the Lame's constants, and for any finite function $\varphi(\vec{\xi}, t)\in\Omega\times[0, T)$, and $\langle\varphi\rangle$

denotes its geometric average over Ω with the volume $|\Omega|$:

$$\langle \varphi \rangle = \frac{1}{|\Omega|} \int_{\Omega} \varphi \, d\vec{\xi}. \tag{19}$$

We note that the medium in consideration is heterogeneous. If we endow Ω with the properties of an appropriate functional space, say the Sobolev's Space, we can then understand any operations on it to be in the generalized (or distributional) sense.

Now, assume hyperelasticity for Ω and take the Frechet derivative of W in Eq. (15) with respect to the geometry $\overset{\circ}{\nabla}\vec{R}$. This gives us its energy conjugate stress tensor - Piola's stress $\tilde{P}$:

$$\tilde{P} \equiv \frac{\partial W}{\partial \overset{\circ}{\nabla}\vec{R}} = 2\lambda_2 \overset{\circ}{\nabla}\vec{R} + (\lambda_1 S_1 - 2\lambda)\tilde{O} + 2\lambda_o \vec{c}\vec{c} \cdot \overset{\circ}{\nabla}\vec{R}. \tag{20}$$

This is the constitutive law for the medium. Then, the motion equation is

$$\overset{\circ}{\nabla} \cdot \tilde{P} = \rho_h \frac{\partial^2 \vec{R}}{\partial t^2}, \tag{21}$$

where $\rho_h = \langle \rho_o \rangle$ is the effective density of the medium in the initial configuration.

The component form of Eq. (21), using Eqs. (11), (12), (14) and (20), gives the following wave equations:

$$c_L^2 \left\{ \frac{\partial}{\partial r}\left[\frac{\chi^2}{r} \frac{\partial}{\partial r}(rf) \right] - \chi^2 f \left(\frac{\partial \theta}{\partial r}\right)^2 \right\} + \frac{\lambda_o}{\rho_h} \frac{f}{r^2} = \frac{\partial^2 f}{\partial t^2} - f\left(\frac{\partial \theta}{\partial t}\right)^2, \tag{22}$$

$$c_L^2 \left[\frac{\partial}{\partial r}\left(\chi^2 f \frac{\partial \theta}{\partial r}\right) + \frac{\chi^2}{r} \frac{\partial}{\partial r}(rf) \frac{\partial \theta}{\partial r} \right] = f \frac{\partial^2 \theta}{\partial t^2} + 2 \frac{\partial f}{\partial t} \frac{\partial \theta}{\partial t}, \tag{23}$$

$$q = g\chi^2, \tag{24}$$

where

$$\chi = \sqrt{1 - \frac{\kappa}{D}}, \tag{25}$$

and

$$c_L^2 \equiv \frac{g}{\rho_h}, \qquad g \equiv \lambda_o + \lambda_1 + 2\lambda_2, \qquad \kappa \equiv \frac{2(\lambda_1 + \lambda_2)}{\lambda_o + \lambda_1 + 2\lambda_2}. \tag{26}$$

3.2. Effect of anisotropy and method of solution

If the medium Ω is homogeneous, then by Eq. (18) we obtain from Eqs. (24) and (26) that

$q \to q_o$, $g \to g_o$, $c_L \to c^o_L$, $\rho_h \to \rho_o$, $\chi \to \chi_o$, $\kappa \to \kappa_o$ and

$$q_o = g_o \chi_o, \quad c_L^o \equiv \sqrt{\frac{g_o}{\rho_o}}, \quad g_o = \lambda + 2\mu, \quad \kappa_o = \frac{2(\lambda+\mu)}{\lambda+2\mu} = \frac{1}{1-\nu}, \tag{27}$$

where ν, $\nu \in$ (-1, 1/2) is the Poisson's ratio.

In Eq. (22) we recognize $\frac{\lambda_o}{\rho_h}\frac{f}{r^2}$ as the anisotropic term and therefore $\frac{\lambda_o}{\rho_o} \equiv \beta$ as the *parameter of anisotropy*. Then $f(r, t)$ is the function carrying the direct effect of anisotropy.

Now we put the following parameters in their dimensionless form:

$$\beta_o \equiv \frac{\beta}{c_L^2}, \quad u \equiv \frac{f}{r_o}, \quad x \equiv \frac{r}{r_o}, \quad \tau \equiv \frac{c_L}{r_o} t, \tag{28}$$

and generate u in β_o:

$$u(x,\tau) = \sum_{n=0}^{\infty} \beta_o^n u_n(x,\tau). \tag{29}$$

Then inserting Eq. (29) in Eq. (22) and Eq. (23), we obtain the recurrence system of equations with respect to powers of the anisotropic parameter β_o:

$$\sum_{n=0}^{\infty} \beta_o^n (X_n^k - T_n^k + F_{n-1}^k) = 0, \ k = 1,2 \tag{30}$$

where,

$$F_{n-1}^k = F_{n-1}^k(u_n); \quad F_{n-1}^k = 0 \ \text{ if either } \ n-1<0 \ \text{ or } \ k=2 \ \text{ and } \ F_{n-1}^1 = \frac{u_{n-1}}{x^2};$$

$$X_n^1 \equiv \frac{\partial}{\partial x}\left(\frac{\chi^2}{x}\frac{\partial}{\partial x}(x u_n)\right) - \chi^2 u_n \left(\frac{\partial\theta}{\partial x}\right)^2, \quad T_n^1 \equiv \frac{\partial^2 u_n}{\partial \tau^2} - u_n\left(\frac{\partial\theta}{\partial\tau}\right)^2;$$

$$X_n^2 \equiv \frac{\partial}{\partial x}\left(\chi^2 u_n \frac{\partial\theta}{\partial x}\right) + \frac{\chi^2}{x}\frac{\partial}{\partial x}(x u_n)\frac{\partial\theta}{\partial x}, \quad T_n^2 \equiv u_n \frac{\partial^2\theta}{\partial\tau^2} + 2\frac{\partial u_n}{\partial\tau}\frac{\partial\theta}{\partial\tau}.$$

The first system in Eq. (30), corresponding to n=0, being the coefficient in zero power of β_o, does not contain the anisotropic term any longer:

$$\frac{\partial}{\partial x}\left[\frac{\chi^2}{x}\ \frac{\partial}{\partial x}(xu_o)\right]-\chi^2 u_o\left(\frac{\partial\theta}{\partial x}\right)^2=\frac{\partial^2 u_o}{\partial\tau^2}-u_o\left(\frac{\partial\theta}{\partial\tau}\right)^2,$$

$$\frac{\partial}{\partial x}\left(\chi^2 u_o\frac{\partial\theta}{\partial x}\right)+\frac{\chi^2}{x}\ \frac{\partial}{\partial x}(xu_o)\frac{\partial\theta}{\partial x}=u_o\frac{\partial^2\theta}{\partial\tau^2}+2\frac{\partial u_o}{\partial\tau}\ \frac{\partial\theta}{\partial\tau}. \tag{31}$$

These expressions coincide with the structure in the case of a homogeneous medium. Even, in the subsequent systems of Eq. (30), when n>0, the anisotropic term now features as a “fictitious force”, being known from the solution of the previous system.

In this way, the problem associated with anisotropy of the medium is completely exposed; and the issues of dispersion and or filteration can then be studied by taking into account the recurrence structure of Eq. (30). We are now left with the problem of finite deformation, with respect to the possible types of waves that propagate in the medium.

4. The effect of finite deformation on wave speed

4.1. Method of characteristics

We associate with $u\ (x,\ \tau)$ longitudinal waves propagating with the velocity c_L given by Eq. (26), by analogy with waves for infinitesimal small deformations Eq. (5), while we associate with $\theta(x,\ \tau)$ shear waves. From system Eq. (31) it is clear that these waves are coupled. To invoke the characteristic method we rewrite that system in a more suitable form:

$$\frac{\partial^2 v}{\partial\tau^2}=a_{11}\frac{\partial^2 v}{\partial x^2}+a_{12}\frac{\partial^2\theta}{\partial x^2}+a_1,$$

$$\frac{\partial^2\theta}{\partial\tau^2}=a_{21}\frac{\partial^2 v}{\partial x^2}+a_{22}\frac{\partial^2\theta}{\partial x^2}+a_2. \tag{32}$$

where

$$v\equiv u_o,\qquad \frac{1}{x}\frac{\partial}{\partial x}(xv)\equiv D\ \cos\ \omega,\qquad v\frac{\partial\theta}{\partial x}\equiv D\ \sin\ \omega, \tag{33}$$

and

$$a_{11} = \chi^2 + \frac{\kappa}{D}\cos^2\omega = 1 - \frac{\kappa}{D}\sin^2\omega,$$

$$a_{12} = \frac{\kappa}{D^2}\left[\frac{1}{x}\ \frac{\partial}{\partial x}(xv)\right]v\ \sin\omega = \frac{\kappa}{2D}v\ \sin\ 2\omega,$$

$$a_{21} = \frac{\kappa}{D^2}\ \frac{\partial\theta}{\partial x} = \frac{\kappa}{2D}\ \frac{\sin\ 2\omega}{v},$$

$$a_{22} = \chi^2 + v\frac{\partial\theta}{\partial x}\ \frac{\kappa}{D^2}\sin\omega = 1 - \frac{\kappa}{D}\cos^2\omega, \tag{34}$$

$$a_1 = v\left(\frac{\partial\theta}{\partial\tau}\right)^2 - \chi^2 v\left(\frac{\partial\theta}{\partial x}\right)^2 + a_{11}\frac{\partial}{\partial x}\left(\frac{v}{x}\right) + a_{12}\frac{1}{v}\ \frac{\partial v}{\partial x}\ \frac{\partial\theta}{\partial x},$$

$$a_2 = \chi^2\cos\omega\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{1}{v}\left(a_{22}\frac{\partial v}{\partial x}\ \frac{\partial\theta}{\partial x} - 2\frac{\partial v}{\partial\tau}\ \frac{\partial\theta}{\partial\tau}\right) + a_{21}\frac{\partial}{\partial x}\left(\frac{v}{x}\right).$$

Now Eq. (32) constitutes a system of quasi-linear differential equations of order 2, with respect to the independent variables x and τ, for the waves v and θ. Then the method of characteristics is applicable. Let

$$v_r = \frac{\partial v}{\partial\tau},\quad v_x = \frac{\partial v}{\partial x},\quad \theta_r = \frac{\partial\theta}{\partial\tau},\quad \theta_x = \frac{\partial\theta}{\partial x};\quad \frac{\partial v_r}{\partial x} = \frac{\partial v_x}{\partial\tau},\quad \frac{\partial\theta_r}{\partial x} = \frac{\partial\theta_x}{\partial\tau}, \tag{35}$$

and consider the variation of the derivatives of v and θ along some curve C, with a constant slope c, in the plane $\pi(x,\ \tau)$:

$$\frac{dx}{d\tau} = c(v,\theta). \tag{36}$$

Using the condition for hyperbolicity of Eq. (32) (Whitham, 1974), in view of Eqs. (35) and (37), we obtain 4 real values for c:

$$c = \pm 1\ \text{ and }\ c = \pm\chi. \tag{37}$$

4.2. Variation of wave speed and parameter of finite deformation

From Eq. (37) we obtain four characteristics: the first pair $c = \pm\ 1$ giving the linear forward

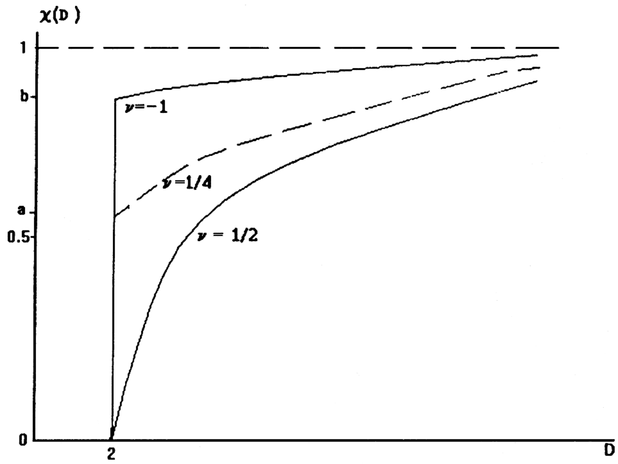

Fig. 1 - Variation of the nonlinear speed χ with respect to the parameter of finite deformation *D*.

and backward wave, while the second pair $c = \pm\chi$ corresponds to the nonlinear Eq. (25) forward and backward wave. In fact,

$$dx = \pm d\tau \quad \text{or} \quad dr = \pm c_L dt; \qquad c_L = \sqrt{\frac{\lambda_o + \lambda_1 + 2\lambda_2}{\rho_h}}.$$

This last expression gives the speed of a longitudinal wave for a layered composite medium and is similar to the case of an homogeneous medium or a small deformation as in Eq. (5). Now in the case of the other pair, we have

$$dx = \pm\chi d\tau, \quad \text{i.e.} \quad dx = \pm\sqrt{(1 - \frac{\kappa}{D})}d\tau. \tag{38}$$

Thus, the 'speed' χ is nonlinear but dependent on *D*, which we refer to as the *parameter of finite deformations*. In the initial configuration $f = r$, $\theta = 0$, and $D = 2$, so that χ becomes a constant linear speed $\chi = \sqrt{\dfrac{\lambda_o + \lambda_2}{\lambda_o + \lambda_1 + 2\lambda_2}}$, and Eq. (38) reduces to

$$dr = \pm c_s dt, \qquad c_s = \sqrt{\frac{\lambda_o + \lambda_2}{\rho_h}},$$

which again coincides with the speed of a transverse cylindrical wave in the case of small deformations for a layered composite, and by Eq. (18) reduces to $c_s = \sqrt{\dfrac{\mu}{\rho_o}}$, as in Eq. (5) for a homogeneous body. On the basis of Eqs. (26) and (27) the variation of χ with respect to *D* is shown in Fig. 1. The continuous curves give the speed variation for $\nu = 1/2$ and $\nu = -1$, respectively. All other materials lie in between.The Poisson's material ($\nu = 1/4$) is given by the broken curve; on the χ-

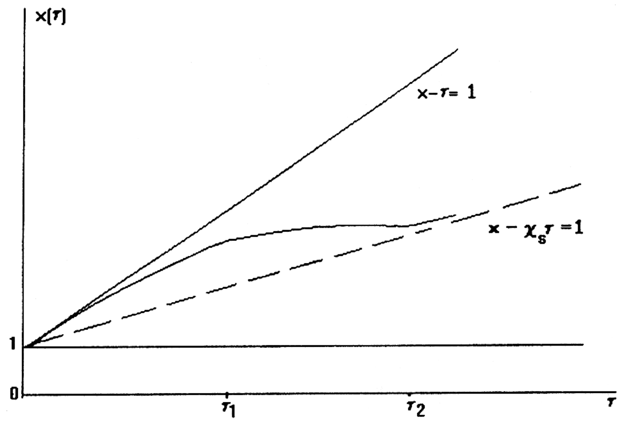

Fig. 2a - Forward characteristics x (τ) for the linear longitudinal wave (continuous line) and the nonlinear coupled wave (continuous curve) which in the limit tends to the linear shear wave (broken line).

axis, a = $1/\sqrt{3}$ and b = $\sqrt{3}/2$.

5. The characteristics curves

The linear characteristics are any straight lines inclined to the temporal axis at an angle $\pi/4$, in the plane $\pi\,(x,\ \tau)$. The main task is establishing the nonlinear ones. First of all, we consider the conditions germane to solving the system (32): for second order equations in 2 variables, we need 8 conditions. We thus consider the following conditions, obtained from physical reasoning.

(i) conditions at initial time $\tau = 0$:

$$v(x,0) = x, \qquad v_\tau(x,0) = 0, \qquad \theta(x,0) = 0, \qquad \theta_\tau(x,0) = 0; \tag{39}$$

(ii) conditions on the internal wall $x = 1$:

$$v(1,\tau) = 1, \qquad \theta(1,\tau) = \varphi(\tau); \tag{40}$$

and

(iii) conditions on the characteristics:

$$v_x(1,0) = 1, \quad \text{on} \quad x - \tau = 1, \tag{41}$$

$$\theta_x(1,0) = -\frac{1}{\chi_i}\frac{d\varphi}{d\tau} \quad \text{on} \;\; dx = \chi(\tau)d\tau, \tag{42}$$

where $\chi_i = \chi(\tau)\big|_{\tau=0} = \chi(0)$, and $\varphi(\tau)$ is an internal excitation.

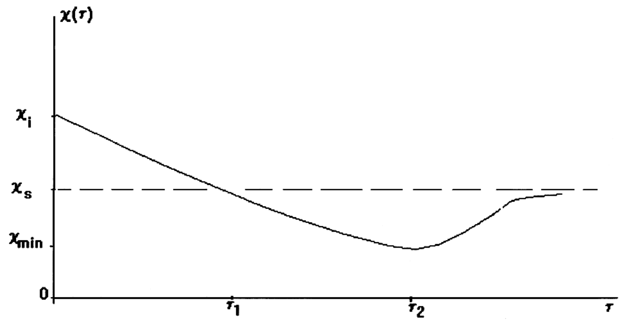

Fig. 2b - Dependence of the nonlinear speed χ on the dimensionless time τ, in the case of a nonzero initial impulse, corresponding to $\alpha \neq 0$ or $\partial\rho\varphi_i \neq 0$. The horizontal broken line corresponds to the constant speed for the linear shear wave.

5.1. Initial propagating speed for the nonlinear wave

Now crucial to obtaining the characteristics is the need to know the initial propagation speed χ_i. On the basis of conditions from Eq. (39) to Eq. (42) and using appropriate jump conditions on v and θ, we observe that $D\,(0) \equiv D_i = \sqrt{\left(4+\theta_x^2\right)}$, and we obtain a bi-cubic equation with respect to χ_i:

$$\chi_i^6 - (2-\alpha)\chi_i^4 + (1-\beta-2\alpha)\chi_i^2 + \alpha = 0, \tag{43}$$

where

$$\alpha \equiv (1/2\partial\varphi_i)^2, \qquad \beta = (\kappa/2)^2. \tag{44}$$

We solve Eq. (43) as a cubic equation with respect to χ_i^2. The pertinent root, which is real and reduces to the speed value χ_s when particles are not excited,

$$\chi_{i|\alpha=0}^2 = \chi_s^2 = 1 - \frac{\kappa}{2}_{|\alpha=0}, \tag{45}$$

is

$$\chi_i^2 = \frac{2-\alpha}{3} - 2\sqrt{\frac{\gamma}{3}}\cos(\pi/3+\psi), \tag{46}$$

where $\gamma = \frac{1}{3}\,(\alpha^2 + 2\alpha + 1 + 3\beta)$ and ψ is just any constant angle. On the basis of Eq. (45) and

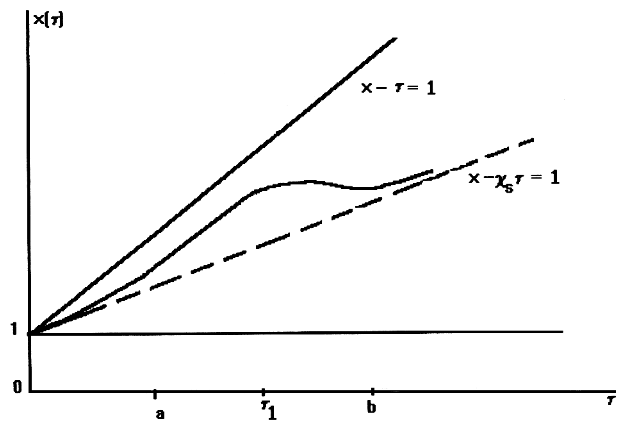

Fig. 3a - The characteristics: the inclined continuous and broken lines correspond to the linear longitudinal and shear waves, respectively, while the curve is for the nonlinear wave when the angular velocity (or impulse) of the initial excitation is zero, and in the limit tends to the linear shear wave $a = \tau_1 (1 - 1/\sqrt{2})$ and $b = \tau_1 (1 + 1/\sqrt{2})$.

Eq. (46) we can now construct the characteristics.

5.2. Construction of the nonlinear characteristic

Assume that at some distance from the source, particles no longer experience excitation and as such are at rest. Then by Eq. (45) the characteristic curve at time $\tau = 0$ is

$$x(\tau) = 1 + \chi_s \tau. \tag{47}$$

At other times, depending on the initial excitation, we can then construct $x(\tau)$ for all τ. In fact, consider any function

$$\eta(\tau); \quad \eta(\tau) \geq 0, \quad |\eta(\tau)| < 1 \;\; \forall \tau \tag{48}$$

depending on the initial excitation, as contained in Eq. (44), then we have

$$x(\tau) = 1 = \chi_s \tau + \eta(\tau). \tag{49}$$

We identify two situations corresponding to $\alpha \neq 0$ and $\alpha = 0$. In those cases we choose respectively

$$\eta(\tau) = p\tau e^{-\frac{\tau}{\tau_1}}; \quad \eta(\tau) = p_o \tau^2 e^{-\frac{2\tau}{\tau_1}}; p > 0, \; p_0 > 0, \tau_1 > 0,$$

and consider the behaviour of $x(0)$, $\chi(\tau)$, and $\dfrac{d^2 x}{d\tau^2} = \dfrac{d\chi}{d\tau}$.

In both cases we obtain oscillatory curves as given in Figs. 2a and 3a, respectively. The

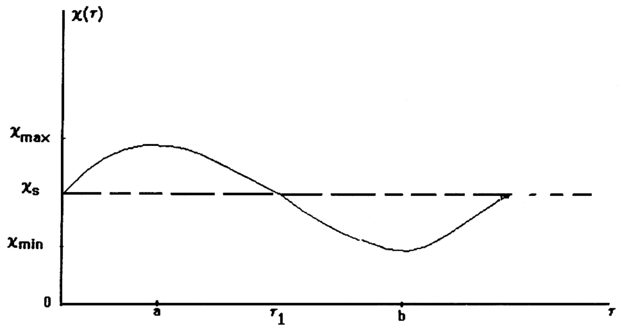

Fig. 3b - Dependence of the nonlinear speed χ on τ, in the case of zero initial impulse $\boldsymbol{a} = \tau_1\ (1 - 1/\sqrt{2})$, while $b = \tau_1\ (1 + 1/\sqrt{2})$.

expression for the curves in the first case is

$$x(\tau) = 1 + \chi_s \tau \left[1 + \left(\frac{\chi_i}{\chi_s} - 1 \right) e^{-\frac{\tau}{\tau_1}} \right], \tag{50}$$

and the minimum point of χ occurs at $\tau = \tau_2 = 2\tau_1$ (see Fig. 2b). Likewise, the expression for the characteristic curves in the second case is

$$x(\tau) = 1 + \chi_s \tau + p_o \tau^2 e^{-\frac{2\tau}{\tau_1}}, \tag{51}$$

where

$$p_o = (\chi_{\max} - \chi_{\min}) \left\{ e^{-2} \tau_1 \left[(\sqrt{2} - 1) e^{\sqrt{2}} + (\sqrt{2} + 1) e^{-\sqrt{2}} \right] \right\}^{-1},$$

and the maximum and minimum values occur respectively at points a and b in Fig. 3b, i.e. $\chi_{\max} = \chi(\mathrm{a}) = \chi(1-1/\sqrt{2})\tau, \chi_{\min} = \chi(\mathrm{b}) = \chi(1+1/\sqrt{2})$.

6. Conclusions

We have used one of the advantages inherent in the method of characteristics, such that even without having explicitly solved boundary value problems, we obtained the pattern of propagation for the cylindrical wave, when finite deformations are accommodated. The obtained characteristics show that locally, the propagating waves interact: one is linear, while the other is nonlinear. What is more, in the limit, the nonlinear wave asymptotically tends to the linear shear wave, already known in the literature. These may have important use in encoding signals between sources and targets in oil exploration and other geophysical applications.

Although a layered elastic medium is considered here, when the assumption of anisotropy/homogeneity is removed, the construction is such that all obtained relations, in view of

Eqs. (18) and (27), reduce to the case of an isotropic/homogeneous medium and as such all the observations made, with respect to finite deformation effects (Lurie, 1980), hold true.

Acknowledgment. This work is partly supported by a grant from the Swedish Agency for Research Cooperation with developing Countries (SAREC) and the International Centre for Theoretical Physics (ICTP). I note with gratitude the interaction with the department of Lithosphere Geophysics, OGS, via seminars.

References

Achenbach J. D.; 1973: *Wave Propagation in Elastic Solids.* North Holland Publishing Company, Amsterdam, 425 pp.

Barclay D. W.; 1998: *Wavefront Expansions for Nonlinear Axial Shear Wave Propagation.* Int. J. Nonlinear Mechanics, **33**, 257-274.

Carcione J. M. and Cavallini F.; 1993: *Energy Balance and Fundamental Relations in Anisotropic-Viscoelastic Media.* Wave Motion, **18**, 11-20.

Carcione J. M. and Seriani G.; 1994: *Torsional Oscillations of Anisotropic Hollow Circular Cylinders.* Acoustic Letters, **18**, 99-102.

Carcione J. M. and Cavallini F.; 1995: *On the Acoustic Electromagnetic Analogy.* Wave Motion, **21**, 149-162.

Carcione J. M. and Quiroga-Goode G.; 1996: *Full Frequency-range Transient Solutio for Compressional Waves in a Fluid-saturated Viscoacoustic Porous Medium.* Geophys Prospecting, **44**, 99-129.

Carcione J. M.; 1997: *Reflection and Transmission of qP-qS Plane Waves at a Plane Boundary Between Viscoelastic Transversely Isotropic Media.* Geophys. J. Int., **129**, 669-680.

Castro R. R., Pacor F. and Petrungar C.; 1997: *Determination of S-wave Energy Release of Earthquakes in the Region of Friuli, Italy.* Geophys. J. Int., **128**, 399-408.

Courant R., Hilbert D.; 1962: *Methods of Mathematical Physics.* Vol. II Interscience Publishers, New York, 830 pp.

Crampin S. and Zatsepin S. V.; 1997: *Modelling the Compliance of Crustal Rock-II. Response to Temporal Changes Before Earthquakes.* Geophys. J. Int., **129**, 495-506.

Green A. E., Adkins J.; 1960: *Large Elastic deformations.* Oxford University Press, 348 pp.

Honarvar F. and Sinclair A. N.; 1996: *Acoustic Wave Scattering from Transversely Isotropic Cylinders.* Acoustic Society of America, **100**, 57-63.

Ibitoye S. A., Akinola A.; 1993: *On Simultaneous Shear-Dilatation Cylindrical Waves in Large Deformation.* Proceedings, Canadian Congress of Applied Mechanics (CANCAM'93), **2**, 491-492.

Igel H. , Ben-Zion Y. and Leary P. C.; 1997: *Simulation of SH- and P-SV-wave Propagation in Fault Zones.* Geophys. J. Int., **128**, 533-546.

Kaduchak G., Wassmuth C. M. and Loeffler C. M.; 1996: *Elastic Wave Contribution in High-resolution Acoustic Images of Fluid-filled, Finite Cylindrical Shells in Water.* The Journal of Acoustic Society of America, **100**, 64-71.

Kohler W., Papanicolaou G. and White B.; 1996; *Localization and Mode Conversion for Elastic Waves in Randomly Layered Media.* Wave Motion, **23**, 1-22.

Lurie A. I.; 1980: *Nonlinear Elasticity.* Nauka Publishers, Moscow, 512 pp. (in Russian)

Lyakhovsky V., Reches Z., Weinberger R. and Scott Thurman E.; 1997: *Nonlinear Elastic Behaviour of damaged Rocks.* Geophys. J. Int., **130**, 157-166.

Pobedria B. E.; 1984; *Mechanics of Composite Materials.* Moscow State University Press, Moscow, 336 pp. (in Russian)

Sujunshkaliev N.; 1980: *A Problem on Large Elastic deformation Theory of Elasticity.* Doklad Academy Nauk UzSSR **5**, 24-28. (in Russian)

Vavryčuk V. and Yomogida K.; 1996: *SH-wave Green Tensor for Homogeneous Transversely Isotropic Media by Higher-*

order Approximation in Asymptotic Ray Theory. Wave Motion, **23**, 83-93.

Whitham G. B.; 1974: *Linear and Nonlinear Waves.* John Wiley & Sons Inc., New York, 636 pp.

Zatsepin S. V. and Crampin S.; 1997: *Modelling the Compliance of Crustal Rock-I. Response of Shear-wave splitting to differential Stress.* Geophys. J. Int., **129**, 477-494.