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Scalar wave field due to a source in a half-space in the presence of a crack

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Abstract. We consider an *SH*-type wave, emitted by a source in a half-space, incident on a plane semi-infinite crack. The incident wave is obtained as a solution of an inhomogeneous wave equation. The diffracted field is found using the Wiener-Hopf technique. The far-field is calculated using asymptotic approximations.

1. Introduction

The problem of diffraction of elastic waves has attracted considerable attention during the last two decades due to its importance in material sciences, seismology and earthquake engineering. In this context Kazi (1975), Asghar and Zaman (1986, 1987) and Zaman and Bokhari (1995) have obtained the diffracted field in the case of a time-harmonic incident wave on plane discontinuities using the Wiener-Hopf technique. This technique was first used by Wiener and Hopf (1931) to solve a singular integral equation. Jones (1952) subsequently presented a modification with which certain mixed boundary value problems could be solved using this procedure without having to obtain an integral equation formulation. In various applications such as acoustic, electromagnetic and ocean waves, Williams (1956), Rawlins (1974) and Hurd (1954) among others, have considered diffraction problems using this method. A good account of this method and many of its applications are presented in Noble (1958). We shall present here a brief outline of the method for the interested reader.

The present paper deals with the diffraction of a horizontally polarized shear (*SH*-) wave from a plane semi-infinite crack. The problem considered differs from our earlier studies mentioned above in the respect that, in the present case, the incident field is produced by the presence of a source emitting a time-harmonic *SH*-type disturbance. Moreover, the far-field approximation is also given at the end.

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2. Wiener-Hopf technique

Wiener and Hopf (1931) presented an excellent technique for solving certain singular integral equations. It was later shown by Copson (1946) that mixed boundary value problems arising from diffraction by a semi-infinite plane could be formulated in terms of integral equations and then solved by the Wiener-Hopf technique. The integral equation was transformed into a functional equation in the complex plane using integral transforms and then solved using analytic continuation argument. In the modified method of Jones (1952), the Fourier transform is directly applied to the partial differential equation without having to obtain an integral equation formulation first. In either of the two approaches, the Fourier or Laplace transforms can be used to reduce the problem to that of solving the functional equation

$$F_+(\zeta)G(\zeta) = R_+(\zeta) + S_-(\zeta), \quad \tau_0 < \operatorname{Im}\zeta < \tau_+, \tag{1}$$

where the subscripts ± refer to functions that are regular in the upper half ζ-plane Im $(\zeta) > \tau_-$ (lower half-plane Im $(\zeta) < \tau_+$). In Eq. (1), the unknown functions are $F_+(\zeta)$ and $S_-(\zeta)$, while $G(\zeta)$ and $R_+(\zeta)$ are known. The first crucial step in solving this equation is to factorize the known function $G(\zeta)$ as $G(\zeta) = G_+(\zeta)G_-(\zeta)$, where $G_+(\zeta)$ is regular in the upper half plane Im $(\zeta) > \tau_-$, while $G_-(\zeta)$ is regular in Im $(\zeta) < \tau_+$. It is further required that $G_+(\zeta)$ and $G_-(\zeta)$ are free of zeros in the respective half-planes of their regularity. Eq. (1) can then be written as

$$F_+(\zeta)G_+(\zeta) = \frac{R_+(\zeta)}{G_-(\zeta)} + \frac{S_-(\zeta)}{G_-(\zeta)}, \quad \tau_- < \operatorname{Im}(\zeta) < \tau_+. \tag{2}$$

We now have the left-hand side of this equation regular in the upper half-plane, and $S_-(\zeta) / G_-(\zeta)$ on the right-hand side regular in the lower half-plane. However, $R_+(\zeta) / G_-(\zeta)$ still is not regular in either of the two half-planes. For this purpose we use the additive decomposition

$$\frac{R_+(\zeta)}{G_-(\zeta)} = M_+(\zeta) + M_-(\zeta), \tag{3}$$

where $M_\pm(\zeta)$ are regular in the upper (lower) half-plane. Thus we now have

$$F_+(\zeta)G_+(\zeta) - M_+(\zeta) = M_-(\zeta) + \frac{S_-(\zeta)}{G_-(\zeta)}, \quad \tau_- < \operatorname{Im}\zeta < \tau_+. \tag{4}$$

Since the left-hand side of Eq. (4) now contains functions that are regular in the upper half-plane, while the right-hand side is regular in the lower half-plane, with the two half-planes overlapping in the strip $\tau_- <$ Im $(\zeta) < \tau_+$, we deduce by analytic continuation that both sides equal an entire function $P(\zeta)$. This entire function can be determined by its behaviour as $|\zeta| \to \infty$. In most problems, we have $P(\zeta) \to 0$ as $|\zeta| \to \infty$. In such a case we find the two unknown functions as

$$\begin{aligned} F_+(\zeta) &= \frac{M_+(\zeta)}{G_+(\zeta)}, \\ S_-(\zeta) &= -M_-(\zeta)G_-(\zeta). \end{aligned} \tag{5}$$

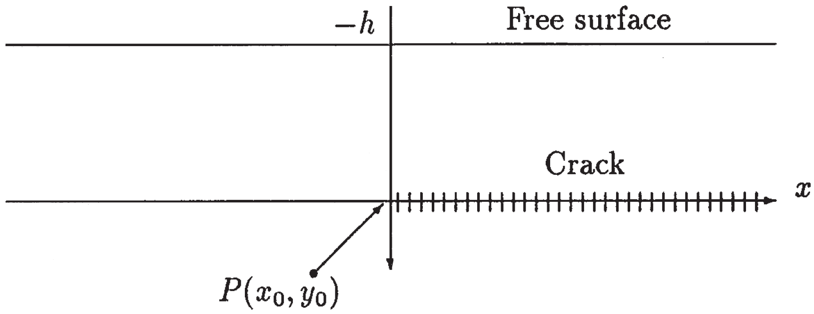

Fig. 1 - The study model.

3. Formulation of the problem

A time-harmonic line source emitting *SH*-type waves is assumed to be situated at $P(x_0, y_0)$ in a homogeneous elastic half-space with free surface $y = -h$ (Fig. 1). The x-axis is chosen along the semi-infinite plane crack ($x > 0$). The rigidity, density and shear slowness of the medium are denoted by μ, ρ, and S_T respectively.

We write the total displacement field in $-\infty < x < \infty$ as

$$w^{tot}(x,y,t) = \begin{cases} w^{inc}(x,y,t) + w_1(x,y,t), & -h < y < 0, \\ w^{inc}(x,y,t) + w_2(x,y,t), & y > 0, \end{cases} \tag{6}$$

where w^{inc} accounts for the inhomogeneous source term, and w_1, w_2 are the diffracted waves in the two regions separated by the crack. The equation of motion satisfied by w^{inc} is

$$\frac{\partial^2 w^{inc}}{\partial x^2} + \frac{\partial^2 w^{inc}}{\partial y^2} - S_T^2 \frac{\partial^2 w^{inc}}{\partial t^2} = \delta(x - x_0)\,\delta(y - y_0), \tag{7}$$

where $k^2 = S_T^2\,\omega^2$, S_T is the slowness of *SH*-wave given by $\sqrt{\rho/\mu}$ and ω is the angular frequency.

The initial conditions are given by

$$w^{inc}(x,y,0) = \frac{\partial w^{inc}(x,y,0)}{\partial t} = 0. \tag{8}$$

The diffracted wave satisfies the homogeneous equation obtained from Eq. (7) by putting the right-hand side zero. The boundary conditions are:

(a) At $x > 0$

$$\left.\begin{aligned} y = 0^-, \ \frac{\partial w_1}{\partial y} \\ y = 0^+, \ \frac{\partial w_2}{\partial y} \end{aligned}\right\} = -\frac{\partial w^{inc}}{\partial y}, \tag{9a}$$

due to the presence of the crack.

(b) At $x < 0$, $y = 0$

$$w_1 = w_2 \,, \tag{9b}$$

and

$$\frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial y} \,, \tag{9c}$$

due to continuity of displacement and stress.
(c) At $y = -h$, $-\infty < x < \infty$

$$\frac{\partial w_1}{\partial y} = 0. \tag{9d}$$

In addition, appropriate edge conditions are imposed for uniqueness of the solution. These conditions specify behaviour of the stress near the edge $x = 0$, and are discussed in Achenbach (1973).

4. Solution of the incident wave

We define the Fourier transform in x as

$$w^*(\zeta, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x, y) e^{i\zeta x}\, dx, \tag{10}$$

with the inversion given by

$$w(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w^*(\zeta, y) e^{-i\zeta x}\, d\zeta, \tag{11}$$

Eq. (7) transforms into

$$\frac{d^2 w^{*inc}(\zeta, y)}{dy^2} - \gamma^2 w^{*inc}(\zeta, y) = \frac{1}{\sqrt{2\pi}} e^{i\zeta x_0} \delta(y - y_0), \tag{12}$$

where $\gamma^2 = \zeta^2 - k^2$, and k is assumed to have a positive imaginary part.

The solution of Eq. (12), together with appropriate boundary conditions for the half-space $y > 0$ (vanishing tractions at the free surface), is given by Noble (1958) as

$$w^{*inc}(\zeta, y) = -\frac{e^{i\zeta x_0}\left\{e^{-\gamma|y-y_0|} + e^{-\gamma|y+y_0|}\right\}}{2\gamma\sqrt{2\pi}}. \tag{13}$$

The inversion formula (11) gives

$$w^{inc}(x, y) = \frac{1}{4i}\left\{H_0^1(k R_1) + H_0^1(k R_2)\right\}, \tag{14}$$

where $R_{1,2}^2 = (x - x_0)^2 + (y \pm y_0)^2$, where the lower sign is taken for the first subscript of R^2, and $H_0^1(kR)$ is the Hankel function of the first kind representing an outgoing wave at infinity. When the source is removed to infinity, the following asymptotic form of the Hankel function gives the far-field behavior:

$$H_0^1(kR) \approx \left(\frac{2}{\pi k R}\right)^{1/2} e^{i(kR - \frac{\pi}{4})}. \tag{15}$$

5. Determination of the diffracted field

We use the Wiener-Hopf procedure to find the solution of the mixed boundary value problem satisfied by the diffracted field.

In order to do so, we decompose $w^*(\zeta, y)$ given by (5) as

$$\begin{aligned} w^*(\zeta, y) &= w_+^*(\zeta, y) + w_-^*(\zeta, y) \\ &= \frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty w(x,y)e^{i\zeta x}\,dx + \int_{-\infty}^0 w(x,y)e^{i\zeta x}\,dx\right\}, \end{aligned} \tag{16}$$

where $\zeta = \alpha + i\tau$. It may be noted that if $|w| < A \exp(\tau_- x)$ as $x \to \infty$, and $|w| < B \exp(\tau_+ x)$ as $x \to -\infty$, then $w_+^*(\zeta, y)$ is analytic for $\tau > \tau_-$, and $w_-^*(\zeta, y)$ is analytic for $\tau < \tau_+$. Thus $w^*(\zeta, y)$ is analytic in the common strip $\tau_- < \tau < \tau_+$. Here we may choose $\tau_\pm = \pm\mathrm{Im}(k)$ (Noble, 1958).

The Fourier transform of the equation of motion satisfied by w_1 and w_2 can be shown to have solutions

$$w_1^*(\zeta, y) = \frac{\cosh\gamma(y+h)}{\gamma\sinh(\gamma h)} w_1^{*'}(\zeta, 0), \tag{17}$$

and

$$w_2^*(\zeta, y) = \frac{-\exp(-\gamma y)}{\gamma} w_2^{*'}(\zeta, 0), \tag{18}$$

where $'$ denotes differentiation with respect to y.

From Eqs. (9a) and (9b) it can be seen that we need to calculate only $w_1^{*'}(\zeta, 0)$ in order to determine $w_1^*(\zeta,, y)$ and $w_2^*(\zeta,, y)$. We therefore need one Wiener-Hopf equation only. This can be obtained from Eqs. (16) and (17), and boundary conditions (9a) and (9b) as

$$w_{1+}^{*'} + (\zeta, 0) + w_{1-}^{*'}(\zeta, 0) = \gamma\exp(-\gamma h)\sinh(\gamma h)\left\{w_{1+}^*(\zeta, 0) - w_{2+}^*(\zeta, 0)\right\}. \tag{19}$$

In order to solve this equation, we use the factorization given by Mittra and Lee (1971) to write

$$\frac{\exp(-\gamma h)\sinh(\gamma h)}{\gamma h} = G_+(\zeta)G_-(\zeta), \tag{20}$$

where $G_-(\zeta) = G_+(-\zeta)$ and

$$G_{\pm}(\zeta) = \left(\frac{\sin kh}{kh}\right)^{1/2} \exp\left[\frac{\pm ih\zeta}{\pi}\left(1 - C + \ln\frac{2\pi}{kh}\right) + \frac{i\pi}{2}\right]$$

$$\exp\left[\frac{i\gamma\ h}{\pi} - \ln\left(\pm\frac{\zeta - \gamma}{\pi}\right)\right]\prod_{n=1}^{\infty}\left(1 \pm \frac{\zeta}{i\gamma nh}\right)\exp\left(\frac{\pm i\zeta h}{n\pi}\right), \tag{21}$$

where C is Euler's constant.

We notice that $G_{\pm}(\zeta)$ have no zeros. Eq. (19) can, therefore, be written as

$$\frac{w_{1+}^{*'}(\zeta,0)}{G_-(\zeta)} + \frac{w_{1-}^{*'}(\zeta,0)}{G_-(\zeta)} = \gamma^2\, hG_+(\zeta)\left\{w_{1+}^{*}(\zeta,0) - w_{2+}^{*}(\zeta,0)\right\}. \tag{22}$$

The additive decomposition theorem (Noble, 1958) can now be utilized to write

$$\frac{w_{1+}^{*'}(\zeta,0)}{G_-(\zeta)} = K_+(\zeta) + K_-(\zeta), \tag{23}$$

where the subscript ± denotes that functions are analytic in the upper (or lower) half-plane, respectively.

The explicit expressions for $K_{\pm}(\zeta)$ are given by the factorization theorem given by Noble (1958) as

$$K_{\pm}(\zeta) = \pm\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{-\mathrm{w}^{*'\,\mathrm{inc}}(\mathrm{g},0)}{\mathrm{G}_-(\mathrm{g})(\mathrm{g}-\mathrm{z})}dg. \tag{24}$$

This integral can easily be evaluated using residue calculus.

We can now write Eq. (18) as

$$K_-(\zeta) + \frac{w_{1-}^{*'}(\zeta,0)}{G_-(\zeta)} = -\,K_+(\zeta) + \gamma^2\, hG_+(\zeta)\left\{w_{1+}^{*}(\zeta,0) - w_{2+}^{*}(\zeta,0)\right\}. \tag{25}$$

Eq. (25) is now in a form such that the left-hand side is regular in the lower half-plane, while the right-hand side in the upper half-plane. As both sides are equal through Eq. (21) in the common strip of analyticity, they define an entire function. As discussed in Achenach (1973), this entire function can be shown to be zero, considering the asymptotic behavior of $w_{1-}^{*'}(\zeta, 0)$ and $K_-(\zeta)$ as $|\zeta|\to\infty$.

Thus we can determine $w_1^{*'}(\zeta, 0)$ as

$$w_{1-}^{*'}(\zeta,0) = -\,K_-(\zeta)G_-(\zeta), \tag{26}$$

where $K_-(\zeta)$ is given by (20), while $G_-(\zeta)$ is known through (21). As noted before, $w_2^{*'}(\zeta, 0)$ can be determined using the value of $w_1^{*'}(\zeta, 0)$.

6. The transmitted waves

We determine the transmitted waves in the two regions separated by the crack using Eqs. (17), (18) and (26), and applying the inverse Fourier transform (11).

a) The region $-h \le y \le 0$; $x > 0$.

$$w_1(x,y) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\cosh\gamma(y+h)}{\gamma\sin(\gamma h)}\left\{w^{*'\,inc}(\zeta,0) + K_-(\zeta)G_-(\zeta)\right\}e^{-i\zeta x}\,d\zeta, \tag{27}$$

where $w^{*'\ inc}(\zeta, 0)$ is known through (13), while $G\_(\zeta)$ and $K\_(\zeta)$ are given by (21) and (24), respectively.

In order to evaluate this integral, we close the contour in the lower half-plane, where $G\_(\zeta)$ and $K\_(\zeta)$ are analytic. The only poles of the integrand are the zeros of sinh γh, apart from the branch point at $\gamma = 0$, which makes an insignificant contribution in the layer (Lapwood, 1949). Now sinh $(\gamma h) = 0$ implies $\gamma = in\pi/h$, $n = 0, 1, 2, ...$, which leads to $\zeta^2 = k^2 - n^2\pi^2/h^2 = -p_n^2$ (say). We, therefore, calculate residues at $\zeta = -ip_n$. The integral in Eq. (27) can thus be evaluated as

$$w_1(x,y) = -i\sqrt{2\pi}\sum\frac{\cosh\gamma_n(y+h)}{\gamma_n\dfrac{d}{d\zeta}\left[\sinh(\gamma h)\right]_{\zeta=-ip_n}}\left[w^{*'\,inc}(ip_n,0) + K_-(-ip_n)G_- - ip_n)\right]e^{-p_n x}. \tag{28}$$

b) The region $y \ge 0$; $x > 0$.

The displacement field in this region can be obtained using Eqs. (13), (18) and (26), and the inversion formula (11) as

$$w_2(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\exp(-\gamma y)}{\gamma}\left\{w^{*'\,inc}(\zeta,0) + K_-(\zeta)G_-(\zeta)\right\}e^{-i\zeta x}d\zeta. \tag{29}$$

In order to evaluate this integral, we again close the contour in the lower half-plane and calculate the branch point contribution. The first term gives rise to a wave of exactly the same form as the incident wave, but with opposite sign, and hence cancels with it. The second term can be evaluated by deforming the contour into a hyperbola by putting $\zeta = -k\cos(\theta + i\tau)$. We estimate the integral for large kr asymptotically and use the asymptotic formula for the Hankel function to write the far-field solution as

$$w_2(x,y) = -\frac{1}{\pi\sqrt{k\gamma}}K_-(-k\cos\theta)G_-(-k\cos\theta)\exp\left\{i\left(kr - \frac{\pi}{4}\right)\right\}, \tag{30}$$

where $x = r\cos\theta$ and $y = r\sin\theta$.

7. Discussion of the results

The waves propagating in the layer $-h \le y \le 0$, $x > 0$ satisfy $k^2 - \zeta^2 = n^2\pi^2/h^2$, which is the dispersion equation for *SH*-waves travelling in a strip of uniform thickness h, with free upper and lower surfaces at $y = -h$ and $y = 0$, respectively. This agrees with the physical situation of the problem, as a layer of uniform thickness is indeed formed due to the presence of crack at $y = 0$ for $x > 0$. The waves in the lower stratum, $y \geq 0$, $x > 0$, are body waves that emanate from the crack acting as a line source, and have the desired behaviour as $r \rightarrow \infty$, because k has been assumed to have a positive imaginary part.

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