The effect of a higher-order nonlinear term on the shallow water finite amplitude wave

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Abstract. A higher-order nonlinear term is introduced into the Korteweg-de Vries (KDV) equation during the process of its formulation from the original shallow-water wave equations. The resultant effects on the fundamental solutions of the equation are described. It is suggested that this theory can satisfactorily model the evolution of swell propagating on beaches with constant depth distribution.

1. Introduction

The evolution of the solitary and other fairly long periodic waveforms on shallow-water of moderate depths is adequately described by the solution of the KDV equation (Zabusky and Galvin, 1971; Segur, 1973; Whitham, 1973). The phenomena has been extensively discussed from variuous aspects in recent years (Whitham, 1973). In this topic, nonlinearity and dispersion dominate, whilst dissipation is usually neglected (Okeke, 1991). Further, the local depth h_0 of the the undisturbed water layer is assumed to be greater than the wave amplitude η_0, defining the peak. Analytically, this is to ensure that the binomial expansion arising from the formulation of the KDV equation converges. Consequently, for oceanographic considerations, this assumption is realistic, since the finite amplitude non-breaking shallow-water satisfy the inequality $\eta_0 < 0.8\ h_0$.

Furthermore, the KDV equation combines the first-order correction in η_0/h_0 and $(h_0/L_0)^2$. Here, L_0 is the horizontal wavelength scale, with $2\pi/L_0$ as the peak wavenumber. Also, higher-order corrections have been extensively discussed (Zabusky and Galvin, 1971; Segur, 1973). However, as a follow up, this model introduces a nonlinear term of order $(\eta_0/h_0)^2$ into the KDV equation in the process of its formulation from the shallow-water wave equations. The effects of this term in the solutions are investigated. An attempt is also made to utilise the model in describing some aspects of finite amplitude shallow-water waves.

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2. The wave equations for shallow water

The model is such that the x-axis is normal to the shoreline, the y-axis being the perpendicular coordinate. The horizontal bottom of the water layer takes the form $y = h_0$, h_0 being constant; $t > 0$ represents the time. The elevation and depression of the free surface are defined by $y = \eta(x,t)$. By assuming that the flow velocity $u(x,t)$ is uniform with depth, the usual shallow-water equations are (Stoker, 1957; Okeke, 1991)

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[u\left(h_0 + \eta\right)\right] = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = 0,$$

(g being the accelaration due to gravity). Following Nettel (1992), $u(x,t)$ takes the form (Whitham, 1973; Nettel, 1992)

$$u(x,t) = 2\sqrt{g(h_0 + \eta)} - 2\sqrt{gh_0}\,. \tag{2}$$

Using Eq. (3) to eliminate $u(x,t)$ in Eqs. (1) or (2), then

$$\frac{\partial \eta}{\partial t} + c_0\left(1 + \frac{3}{2}\frac{\eta}{h_0} - \frac{3}{8}\frac{\eta^2}{h_0^2}\right)\frac{\partial \eta}{\partial x} = O\left(\frac{\eta^3}{h_0^3}\right). \tag{3}$$

Incorporating the dispersion effect of order $(h_0/L_0)^2$ into Eq. (4), then

$$\frac{\partial \eta}{\partial t} + c_0\left(1 + \frac{3}{2}\frac{\eta}{h_0} - \frac{3}{8}\frac{\eta^2}{h_0^2}\right)\frac{\partial \eta}{\partial x} + \beta\frac{\partial^3 \eta}{\partial x^3} = 0, \tag{4}$$

where

$$c_0 = \sqrt{gh_0} \tag{5}$$

and

$$\beta = \frac{1}{6}c_0 h_0.$$

Eq. (5) is a form of the KDV equation with an additional higher-order term involving $(\eta/h_0)^2$.

Development of Eq. (5) in the form of uniform wave trains is obtained by means of a change of variables. Thus,

$$\theta = x - u_0\,t, \qquad u_0^2 = g\left(h_0 + \eta_0\right), \qquad \eta = \eta(\theta)$$

The first integration of Eq. (5) gives

$$-u_0\eta + c_0\left(\eta + \frac{3}{4h_0}\eta^2 - \frac{1}{8h_0^2}\eta^3\right) + \beta\frac{d^2\eta}{d^2\theta^2} = X_0. \tag{6}$$

Eq. (6) can be further integrated to obtain

$$-\frac{u_0}{2}\eta^2 + c_0\left(\frac{\eta^2}{2} + \frac{\eta^3}{4h_0} - \frac{\eta^4}{32h_0^2}\right) + \frac{\beta}{2}\left(\frac{d\eta}{d\theta}\right)^2 = X_0\eta - X_1. \tag{7}$$

X_0 and X_1 are arbitrary constants. They are determined if h_0 is suitably chosen such that

$$\frac{d\eta}{d\theta} = \frac{d^2\eta}{d\theta^2} = 0,$$

where $\eta = a_0$; a_0 being a typical wave amplitude at the seaward edge of the shallow water. Thus,

$$X_0 = \frac{c_0 a_0^2}{4h_0}\left(3 - \frac{a_0}{2h_0}\right) - c_0 a_0\left(\frac{u_0}{c_0} - 1\right) = c_0 X_{00}$$

and

$$X_1 = \frac{a_0^2 c_0}{2}\left(\frac{u_0}{c_0} - 1\right) - \frac{c_0 a_0^3}{2h_0}\left(1 - \frac{3a_0}{16h_0}\right) = c_0 X_{11}.$$

Now, Eq. (7) takes the standard form

$$\gamma\left(\frac{d\eta}{d\theta}\right)^4 = \eta^4 - \alpha_1\eta^3 + \alpha_2\eta^2 + \alpha_3\eta - \alpha_4, \tag{8}$$

where

$$\gamma = 16h_0^2\left(\frac{\beta}{c_0}\right), \quad \alpha_1 = 8h_0, \quad \alpha_2 = 16h_0^2\left(\frac{u_0}{c_0} - 1\right), \quad \alpha_3 = 32h_0^2 X_{00},$$

$$\alpha_4 = 32_0^2 X_{11} \text{ and } \frac{u_0}{c_0} - 1 = \frac{\eta_0}{2h_0} - \frac{\eta_0^2}{8h_0^2} + O\left(\frac{\eta_0^3}{h_0^3}\right).$$

2.1. The associated energy equation

Eq. (7) may be written in the form

$$-\left(\frac{u_0}{c_0} - 1\right)\eta^2 + \left(\frac{\eta^3}{2h_0} - \frac{\eta^4}{16h_0^2}\right) + \frac{h_0^2\dot{\eta}}{16u_0^2} = X_0\eta - X_1. \tag{9}$$

With $\eta_0 = 0.774\ h_0$, u_0/c_0 - 1 ~ 1/3 then Eq. (9) simplifies to

$$-g\eta^2 + g\left(1.5\frac{\eta^3}{h_0} - 0.19\frac{\eta^4}{h_0^2}\right) + \frac{h_0\dot{\eta}^2}{3} = 3\left(X_0\eta - X_1\right). \tag{10}$$

In Eqs. (9) and (10), over dot represents differentiation with respect to time.

Multiply Eq. (10) by the constant density ρ and carry out vertical averaging of the terms in the equation to obtain

$$\rho g\overline{\eta^2} = \frac{\rho h_0}{3}\overline{\dot{\eta}^2} + \rho g\left(1.5\frac{\overline{\eta^3}}{h_0} - 0.19\frac{\overline{\eta^4}}{h_0^2}\right) + X_{12}. \tag{11}$$

X_{12} is a constant that includes the contribution from the mean sea surface elevation $3\bar{\eta}\rho X_0$. In nonlinear wave theory, $\bar{\eta}$ is not necessarily zero. Again, $\dot{\eta}$ is the speed of the free surface elevation $\eta(x,t)$; hence $\dot{\eta}^2$ is proportional to the kinetic energy associated with the free surface oscillations. Further, $\bar{\eta}^2$ is the mean square displacement of the sea surface elevation and thus, proportional to the mean potential energy of the elevation. Thus, Eq. (11) models the energy equation for the shallow-water nonlinear waves when the water depth h_0 is constant.

The term in $\bar{\eta}^4$ suggests the effect of the additional term of order $(\eta/h_0)^2$ in Eq. (5). With $\eta_0 = 0.774\ h_0$ for wave with finite amplitude, the contribution of this term to the total energy of the wave process as expressed by Eq. (11) is estimated to be 8%. This is quite significant percentage which can effect any calculation. Again, the negative sign before the term seems to suggest also its diminishing effect on the potential energy of the system.

3. Solitary wave solution of Eq. (8)

In this case, it is usually assumed that the oscillations vanish at infinity (Whitham, 1973); i.e., $a_0 = 0$; thus 2.8 assumes a simple form:

$$\gamma\left(\frac{d\eta}{d\theta}\right)^2 = \eta^2\left(\eta^2 - \alpha_1\eta + \alpha_2\right). \tag{12}$$

The solution to Eq. (12) is now expressed in terms of elementary functions. In this case, take

$$\eta^2\left(\eta^2 - \alpha_1\eta + \alpha_2\right) = \eta^2(\eta - b)(\eta - d),$$

where $b + d = \alpha_1$ and $bd = \alpha_2$ from which

$$d = 4h_0\left[1 \pm \left(2 - \frac{u_0}{c_0}\right)^{1/2}\right].$$

Separating variables in Eq. (12) and integrating, then,

$$\frac{\theta}{\gamma} + B_0 = \int \frac{1}{\eta\left(\eta^2 - \alpha_1\eta + \alpha_2\right)^{1/2}} d\eta = \int \frac{1}{\eta\left[(\eta - b)(\eta - d)\right]^{1/2}} d\eta,$$

where β_0 is an arbitrary constant which is taken as zero. From which

$$\eta(\theta) = \frac{\alpha_2 \sec h^2(R_1/2)}{d \sec h^2(R_1/2) + A_0} \tag{13}$$

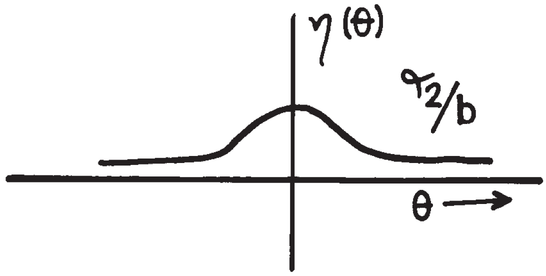

Fig. 1 - A solitary wave form.

$$A_0 = b - d = 8h_0\sqrt{2 - u_0 / c_0} \text{ and } R_1 = 2\left[\sqrt{\alpha_2}\,(\theta / \gamma)\right].$$

As $\theta \to 0$, sech $(R_1/2) \to 1$ and $\eta \to \alpha_2/b$. Again, as $\theta \to \infty$, sech $(R_1/2) \to 0$ and $\eta \to 0$. Thus, even though (13) does not possess the functional form of the familiar solution of the KDV equation (Zabusky and Galvin, 1971; Segur, 1973), it still models a solitary wave with amplitude α_2/b. Hence, solution (13) suggests the effect of an additional higher-order term in Eq. (5) (Fig. 1).

In geophysical applications, consider the non-breaking finite amplitude solitary wave with peak height η_0. Usually, $0.5\ h_0 \le \eta_0 \le 0.8\ h_0$ for local depth h_0. Consequently, if $\eta_0 = 0.5\ h_0$, we have

$$\frac{u_0}{c_0} = \left(1 + \frac{\eta_0}{h_0}\right)^{1/2} = 1.23 \text{ and } \alpha_2 = 3.68 h_0^2,$$

and $b = 7.10\ h_0$ or $b = 0.9\ h_0$. If $\eta_0 = 0.8\ h_0$ then $\mu_0/c_0 = 1.342$, $\alpha_2 = 5.47\ h_0$ and $b = 7.2\ h_0$ or $b = 0.8\ h_0$. This suggests that Eq. (13) can model a non-breaking solitary wave (Kinsman, 1965; Okeke, 1991), since A_0, α_2 and b are always non-negative (Fig. 1).

4. Periodic wave solution

In this consideration, the solution of Eq. (7) does not need to vanish at the seaward end of the shallow water. Thus, the complete equation will now be utilised. Firstly, let $\eta(\theta)=h_0\xi$ and $\xi=\xi(\theta)$. In terms of ξ, Eq. (7) takes the form

$$\gamma h_0^2 = \left(\frac{d\xi}{d\theta}\right)^2 = h_0^4\xi^4 - \alpha_1 h_0^3\xi^3 + \alpha_2 h_0^2\xi^2 + \alpha_3 h_0\xi - \alpha_4. \tag{14}$$

Also, transform from $\xi(\theta)$ to $Z(\theta)$ using

$$\xi = \frac{\lambda Z + \mu}{Z + 2} \tag{15}$$

where λ and μ are constants chosen such that coefficients of the terms involving Z and Z^3 vanish separately when Eq. (15) is substituted into Eq. (14). In this consideration, the vanishing of the coefficient of the terms in Z determines μ in terms of λ. That is,

$$H_0 = \frac{2\left(\alpha_1 H^3 - 2\alpha_2 H^2 - 3\alpha_3 H + 4\alpha_4\right)}{4H^3 - 3\alpha_1 H^2 + 2\alpha_2 H + \alpha_3}, \tag{16}$$

where $H_0 = h_0\,\mu$ and $H = h_0\,\lambda$. Again, using Eq. (16) and the property that the coefficient of the term in Z^3 vanishes, H satisfies the cubic equation

$$\alpha_1 H^3 - 2\alpha_2 H^2 - 3\alpha_3 H + 4\alpha_4 = 0. \tag{17}$$

Thus, Eqs. (16) and (17) imply that $\mu = 0$, and that Z satisfies the standard equation

$$\left(\frac{dZ}{d\theta}\right)^2 = \omega^2\left(1 - Z^2\right)\left(1 - k^2 Z^2\right), \tag{18}$$

where

$$k^2 = \frac{1}{16\alpha_4}\left(\alpha_4 - \alpha_3 H - \alpha_2 H^2 + \alpha_1 H^3 - H^4\right), \tag{19}$$

and

$$\omega^2 = 16\alpha_4 / \gamma_0$$

where

$$\gamma_0 = \gamma H^2.$$

Eq. (18) has a solution

$$Z = Sn\omega\theta$$

$$\xi = \frac{\lambda Sn\omega\theta}{2 + Sn\omega\theta}$$

It thus follows that Eq. (5) has closed solution given by

$$\eta = \frac{H\left[Sn\omega\left(x - u_0 t\right)\right]}{2 + Sn\omega\left(x - u_0 t\right)} \tag{20}$$

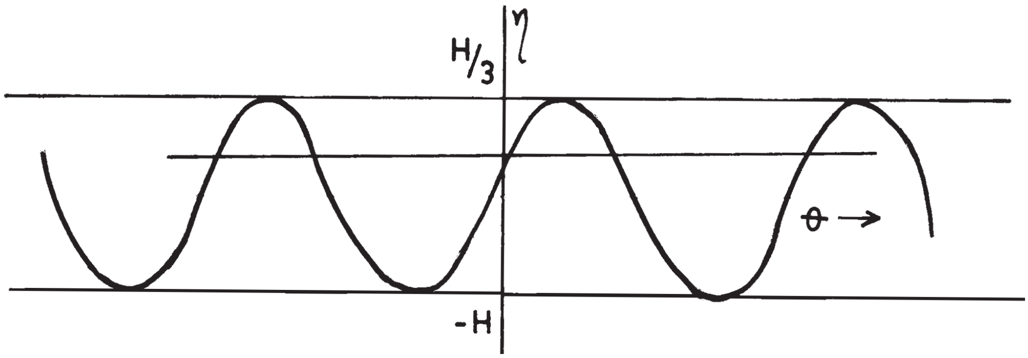

Fig. 2 - A periodic wave form.

The function *Sn* is the Jacobian elliptic function of the first kind. Thus, Eq. (20) represents the *Sn* oscillation with period T; k is the modulus of the elliptic function with γ as the modular angle, where $k = \sin(\gamma)$. T is defined by $T = 4K(k^2)/\omega$. Here, $K(k^2)$ is the complete Jacobian elliptic function of the first kind given by $K(k^2) = \pi\ /2\ {}_2F_1(1/2, 1/2; 1; k^2)$, i.e., in terms of the hypergeometric function ${}_2F_1(1/2, 1/2; 1; k^2)$. Using the properties of the function $K(k^2) = (\pi/2)(1+k^2/4) + O(k^4)$, consequently, $T = (2\pi/\omega)(1 + k^2/4) + O(k^4)$. It should be mentioned that the appearance of the number 2 in the denominator of Eq. (15) and subsequently in Eq. (20) is the outcome of a series of numerical experiments arising from this model.

5. Discussions

In the calculations that follow, length scales are in metres and time scales in seconds. Thus, consider a wave with unit amplitude at the seaward edge of the shallow water, propagating into the beach without breaking. Take h_0 = 2.5 m as the constant depth of the shallow water, corresponding to a wave speed of 4.95 ms^{-1}. With these data, the cubic Eq. (17) is solved numerically, and with error (in its roots) of order 10^{-9}, the general variation of $H(\eta_0/h_0)$ is shown in Table 1. The corresponding variations in k and θ are also shown. In all, $0 < k < 1$ and the numerical

Table 1 - The variations in wave parameters as functions of relative wave height.

η/h_0	H (m)	k	θ^o	T (s)	L (m)
0.8	4.497	0.5301818	32.018	6.284	41.734
0.7	4.474	0.5287870	31.923	6.074	39.203
0.6	4.423	0.5257480	31.719	5.930	37.130
0.5	4.376	0.5229540	31.531	5.793	35.118
0.4	4.331	0.5202930	31.352	5.660	33.148

values of k are sufficiently small to give rapid convergence of the hypergeometric series associated with the function T. Thus, Eq. (20) models a periodic oscillation with period T. However, it is not cnoidal in form, as is usually associated with the solution of the KDV equation (Nettel, 1992). It is further noted that the solution (20) is not very symmetric about the line θ-axis, as clearly shown in Fig. 2. These are the interesting effects of the additional term of O $[\eta_0/h_0]^2$ in the original formulation of the simple model.

Finally, in the range of the periods usually associated with non-breaking finite amplitude waves, Table 1 suggests that this work models (to a large extent) the evolution of a low level swell. If $\eta_0 \in [0.6\ h_0, 0.8\ h_0]$, the modelling is quite satisfactory, but if $\eta_0 \in [0.4\ h_0, 0.6\ h_0]$, the succsess is about 67%. Nevertheless, further adjustments in the numerical values of h_0 and a_0 by a realistic factor of one and a half suggest an interesting model of high level swell with period of 10 seconds and more.

References

Kinsman B.; 1965: *Wind waves.* Prentice-Hall, Inc, pp. 9-24.

Lamb H.; 1945: *Hydrodynamics.* Cambridge University Press, 425 pp.

Nettel S.; 1992: *Wave Physics.* Springer-Verlag, pp. 181-186.

Okeke E. O.; 1991: *The stability of nonlinear water waves.* Africana Mathematika, **4**, 161-167.

Segur H.; 1973: *Korteweg de-Vries equation and water waves.* J. Fluid Mech., **59**, 720-736.

Stoker J.; 1957: *Water waves.* Interscience Publishers, Inc., 567 pp.

Whitham G. B.; 1973: *Linear and Nonlinear Waves.* John Wiley and Sons, pp. 460-470.

Zabusky N. J. and Galvin C. J.; 1971: *Shallow-Water Waves, the KDV Equation and Solitons.* J. Fluid Mech., **47**, 811-824.