# A simple mathematical formula relating the maximun entropy spectral estimate to the amplitude of a noisy signal; application to geoelectrical recordings

M. CIMINALE and M. LODDO

*Dipartimento di Geologia e Geofisica, Università di Bari, Italy*

(Received October 17, 1994; accepted April 10, 1995)

**Abstract.** In geoelectrical prospecting with commutated currents it is very important to correctly evaluate the amplitude of the useful voltage signal necessary to compute the apparent resistivity. This procedure presents serious difficulties, particularly in highly noise-degraded situations. Various methods of filtering and spectral analysis of voltage recordings have been used in the past. In this research the maximum entropy method (MEM) is used. First we do some computations that allow us to obtain a straightforward formula relating the maximum entropy spectral estimate to the amplitude of the useful signal. Then we apply this mathematical relationship to several theoretical and experimental cases. Comparison of the results with those obtained by other spectral analysis techniques and some practical considerations allow us to conclude that, in general, MEM gives the best mathematical solution to the problem.

## 1. Introduction

Strong telluric noise, the use of low power generators, large spacings, and the presence of very low resistivity formations may cause an increase of the noise level in geoelectrical prospecting. As a consequence, it is more difficult to correctly evaluate the amplitude of the useful voltage signal necessary to compute the apparent resistivity. For instance, this becomes a significant problem when dipole soundings are carried out in geothermal areas. In these cases we deal with very low useful voltage signals (the electric field due to the energizing current decreases as the cube of the distance) and the investigated earth materials (impermeable cover of the geothermal reservoir and evaporitic formations) have a very low resistivity.

Corresponding author: M. Ciminale; Dipartimento di Geologia e Geofisica, Università, Campus Universitario, 70125 Bari, Italy

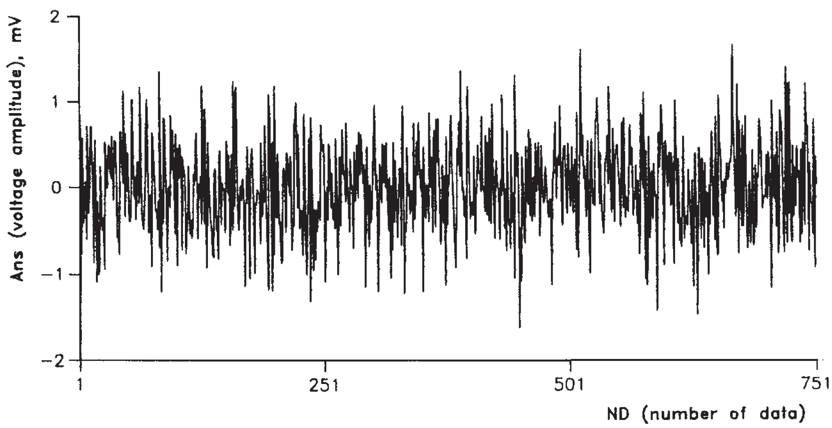


**Fig. 1 -** Theoretical process T1 composed of a sine wave with frequency 1/30 Hz and amplitude .085, and pure white noise. The sampling interval is 1 s. The noise level (defined as the ratio of the standard deviation of the noise to the standard deviation of the signal) is 9.00.

In this paper we leave out the instrumental solution (the use of high power generators) and approach the problem mathematically; that is, we try to resolve and estimate a narrow peak at the working frequency (the frequency of the energizing current) in the general telluric noise. For this purpose, various methods have been studied and used in the past, including maximum likelihood (MLM) (Loddo and Patella, 1978; Ciminale and Patella, 1982), stacking (Alfano et al., 1982; Lapenna et al., 1987) and periodogram (Cuomo et al., 1990).

In this paper the maximum entropy method (MEM) is considered. Authors who have studied MEM in the past (e.g., Ulrych, 1972; Chen and Stegen, 1974; Radosky et al., 1975; Satorious and Zeidler, 1978) concur that this method is far superior in spectral resolution compared to conventional spectral estimation techniques. They stress the accuracy and resolving power of the maximum entropy spectrum when the length of the available noisy data is limited.

With regard to the estimate of the power (or amplitude) of the sinusoidal signal, many scientists (e.g., Ulrych and Bishop, 1975; Radosky et al., 1975; Swingler, 1979) agree with Lacoss (1971) by stating that it is the area of the output peak from MEM which “reflects” or “is proportional to” the input peak. But what do “reflects” or “is proportional to” really mean? None of the papers that have appeared in the literature seems to demonstrate these statements mathematically.

Johnsen and Anderson (1978) avoid numerical integration by suggesting a method for power estimation based on the evaluation of complex residues of the maximum entropy spectral estimator given by Burg (1967).

In the first part of this article, starting from the fundamental paper of Lacoss (1971), a simple alternative procedure (not utilizing any spectral integration) is developed, which allows a straightforward relationship to be obtained between the maximum entropy spectral estimate ($P_e$) and the amplitude of the sinusoidal signal.

In the second part, some applications of this mathematical relationship to theoretical and

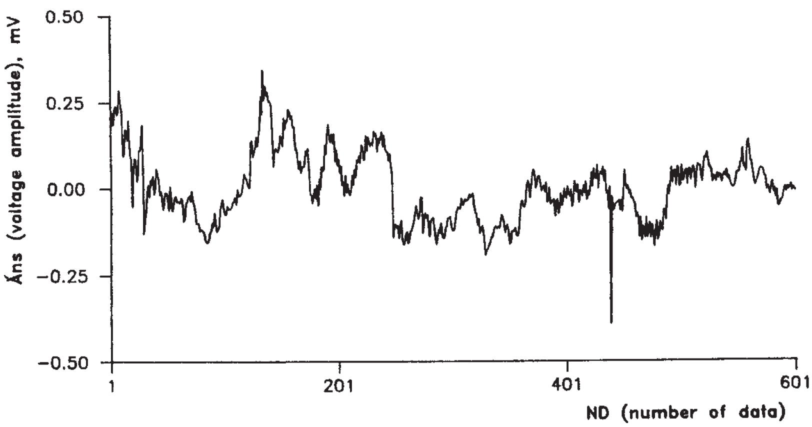


**Fig. 2 -** Semi-theoretical process NS1 composed of a sine wave with frequency 1/30 Hz and amplitude .027, and recorded telluric noise. The sampling interval is 1 s. The noise level (defined in Fig. 1 caption) is 5.10.

experimental cases will be shown and a comparison with other methods of analysis will also be discussed.

## 2. Theoretical background

Suppose that a continuous process under study is composed of a sine wave with power $P_s$ and white noise with unit power. After Lacoss (1971), setting $\phi_s=2\pi f_s\Delta$, where $f_s$ is the frequency of the sine wave in Hz and $\Delta$ is the sampling interval, the peak value of the maximun entropy spectrum $P_e(\phi)$ can be expressed as

$$P_e(\phi_s) = P_s^2\, N^2\, \Delta / 4\ , \tag{1}$$

where $N$ is the number of the autocorrelation function elements for the noisy process being analyzed.

Lacoss (1971) states that eq. (1) is valid if

$$P_s N \gg 1 \text{ and } N \gg 1\ . \tag{2}$$

Moreover, using a Taylor series expansion, the same author suggests an approximate solution for the 3 db width of the peak of the maximum entropy spectral estimate.

We generalize this solution for a generic $n$ dB bandwidth. This extension allows a better approximation to the computations and will have an important practical implication.

The $n$ dB width of the peak is obtained by solving the equation

$$P_e(\phi_n) = 10^{-n/10}\, P_e(\phi_s), \tag{3}$$

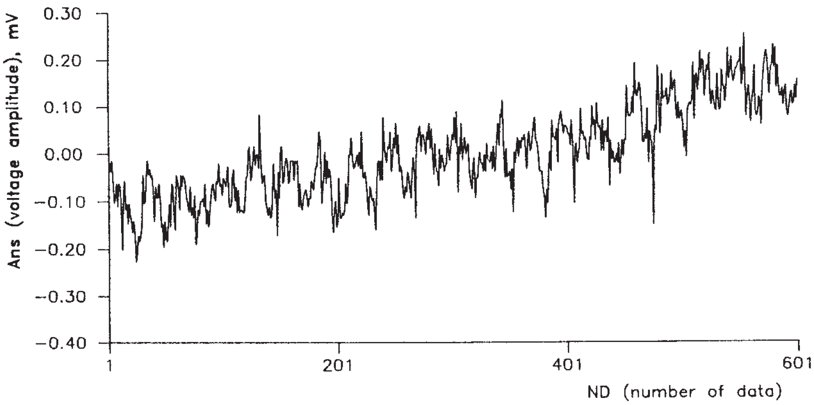


**Fig. 3 -** Complete field recording P7 for an axial sounding. The sampling interval is 1 s. The energizing signal is a current square wave with frequency 1/30 Hz.

where $\phi_n=2\pi f_n \Delta f$ is one of the two frequencies close to $\phi_s$, at which the value of the spectrum has an attenuation of $n$ dB with respect to the peak value. To solve eq. (3), the expression for $P_e(\phi)$ given by Lacoss (1971) for a real correlation function

$$P_e(\phi) = \Delta \left[1 - \frac{P_s}{2+P_s N}\right] / \left|1 - \frac{P_s N}{2+P_s N} B_N^*(\phi_s - \phi)\right|^2, \tag{4}$$

where $B_N(\phi_s\text{-}\phi)=\sum_{n=0}^{N-1} e^{i(\phi_s-\phi)^n}/N$, and * indicates complex conjugation, is used and elaborated.

Eq. (4) can be rewritten in the following form:

$$P_e(\phi_n) = \Delta \left[1 - \frac{P_s}{2+P_s N}\right] / \left[1 - \frac{P_s N}{2+P_s N}\left(B_N(\beta) + B_N^*(\beta)\right) + \left(\frac{P_s}{2+P_s N}\right)^2 B_N(\beta)\, B_N^*(\beta)\right], \tag{5}$$

where

$$\beta = \phi_s - \phi_n \tag{6}$$

indicates the $n$ dB half-bandwidth of the peak.

By using the Taylor series approximation of $B_N(\beta) + B_N^*(\beta)$ and of $B_N(\beta) B_N^*(\beta)$ (Lacoss,1971), and by applying conditions (2), after some manipulation, eq. (5) takes the form

$$P_e(\phi_n) = 4\Delta P_s^2 N^2 / \left(16 + P_s^2 N^4 \beta^2\right). \tag{7}$$

Finally, by replacing eqs. (1) and (7) in eq. (3) and setting $\delta=10^{-n/10}$, we have

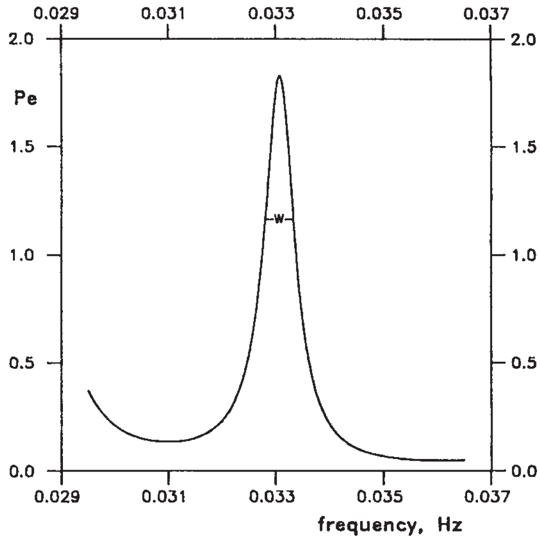


**Fig. 4 -** The maximum entropy power spectrum of the process in Fig. 1. *W* is the 2 dB bandwidth of the peak.

$$16/\left(16 + N^4 P_s^2 \beta^2\right) = \delta \, , \tag{8}$$

and thus

$$\beta = 4\left[(1-\delta)\,/\,\delta\right]^{1/2} / \, P_s N^2 \tag{9}$$

**Table 1 -** Amplitude estimates obtained by the different methods applied without trend removal.
1 Theoretical amplitude (input); 2 Number of data points; 3 Noise level; 4 Amplitude from MEM (output); 5 Amplitude from MLM (output); 6 Amplitude from periodogram (output); 7 Amplitude from stacking (output); 8 Number of coefficients.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| T1 | .0850 | 751 | 9.00 | .0857 | .0895 | .0693 | .1200 | 225 |
| T2 | .0400 | 661 | 6.30 | .0442 | .0430 | .0362 | .0680 | 198 |
| T3 | .5000 | 901 | 4.00 | .5102 | .4833 | .4104 | .5630 | 270 |
| NS1 | .0270 | 601 | 5.10 | .0254 | .0263 | .0225 | .0250 | 180 |
| NS2 | .0200 | 601 | 6.76 | .0249 | .0256 | .0244 | .0290 | 180 |
| NS3 | .0220 | 901 | 22.35 | .0261 | .0346 | .0330 | .0300 | 270 |
| NS4 | .1000 | 601 | 23.36 | .1147 | .1316 | .1389 | .1500 | 180 |
| NS5 | .0850 | 361 | 2.43 | .0849 | .0920 | .0944 | .0950 | 108 |
| NS6 | .0220 | 751 | 17.51 | .0230 | .0272 | .0277 | .0300 | 225 |
| NS7 | .0200 | 601 | 4.93 | .0214 | .0212 | .0184 | .0230 | 180 |
| P5 | - | 601 | - | .0715 | .0690 | .0646 | .0760 | 180 |
| P6 | - | 241 | - | .0846 | .0868 | .0812 | .0900 | 72 |
| P7 | - | 601 | - | .0444 | .0423 | .0398 | .0480 | 180 |

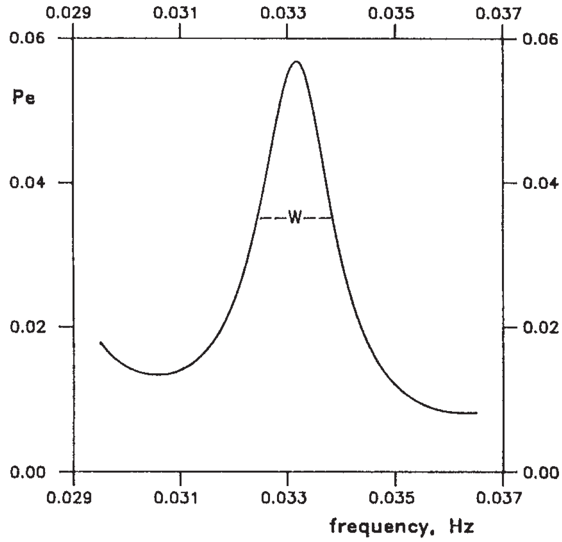


**Fig. 5-** The maximum entropy power spectrum of the process in Fig. 2. *W* is the 2 dB bandwidth of the peak.

By using this equation and the definition (6) we can easily obtain the expression for *W*, the *n* dB bandwidth in Hz. Indeed, setting $W=2(f_s\text{-}f_n)$ and still keeping in mind that $\phi=2\pi f\Delta$, we can write

$$W = 4k\,/\,\pi\Delta P_s N^2 \tag{10}$$

where $k=[(1\text{-}\delta)/\delta]^{1/2}$.
Now the product of peak value (eq. (1) ) with bandwidth (eq. (10) ) is considered:

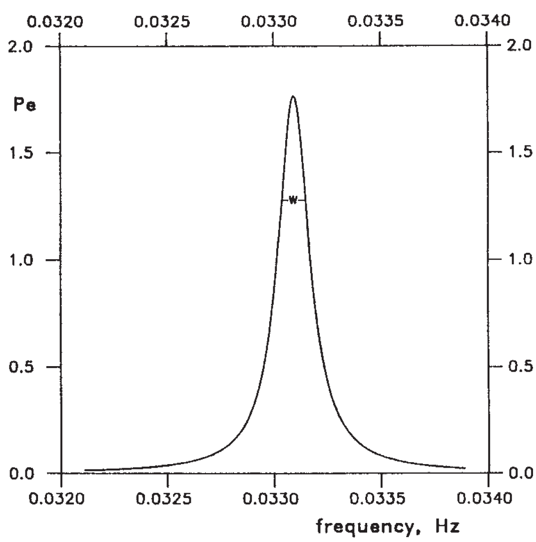


**Fig. 6-** The maximum entropy power spectrum of the process in Fig. 3. *W* is the 2 dB bandwidth of the peak.

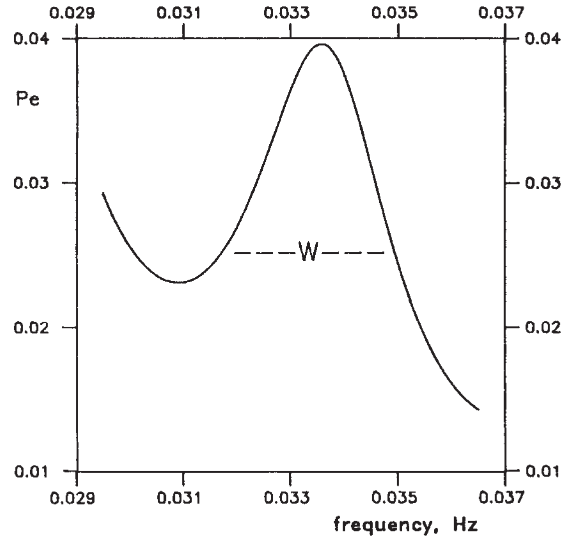


**Fig. 7-** The maximum entropy power spectrum of a semitheoretical process showing a peak not having the 3 dB bandwidth.

$$P_e(\phi_s)\, W = P_s\, k / \pi \tag{11}$$

$$P_s = \pi\, P_e(\phi_s)\, W / k \tag{12}$$

Finally, denoting the amplitude of the sinusoid by A, and since $P_s = A^2/2$, we have from eq. (12)

$$A = \left(2\pi\, P_e(\phi_s)\, W / k\right)^{1/2} \tag{13}$$

Eq. (13) is the simple formula which allows us to obtain the amplitude of the signal straightforwardly from the maximum entropy spectral estimate. In this way, the relationship between $P_e(\phi)$ and $P_s$ claimed by different authors but never proved is now well established. Its validity is bound to conditions (2).

It should be noted that the generalization of the solution to an $n$ dB bandwidth yields a better approximation to the computations. In fact it is possible to choose $n$ in such a way that $\phi_S$ and $\phi_n$ are close enough to give an optimum truncated Taylor series expansion.

## 3. Applications

In geoelectrical prospecting with commutated currents, the voltage time sequence recorded

**Table 2 -** Amplitude estimates obtained by the different methods applied after trend removal.
1 Theoretical amplitude (input); 2 Number of data points: 3 Noise level; 4 Amplitude from MEM (output); 5 Amplitude from MLM (output); 6 Amplitude from periodogram (output); 7 Amplitude from stacking (output); 8 Number of coefficients.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| NS3 | .0220 | 901 | 5.20 | .0250 | .0266 | .0254 | .0260 | 270 |
| NS4 | .1000 | 601 | 14.66 | .1135 | .1089 | .0967 | .1170 | 180 |
| NS5 | .0850 | 361 | 0.49 | .0843 | .0857 | .0816 | .0815 | 108 |
| NS6 | .0220 | 751 | 4.33 | .0226 | .0223 | .0199 | .0190 | 225 |
| NS7 | .0200 | 601 | 3.97 | .0207 | .0223 | .0191 | .0240 | 180 |
| P5 | - | 601 | - | .0715 | .0678 | .0635 | .0740 | 180 |
| P6 | - | 241 | - | .0868 | .0863 | .0834 | .0905 | 72 |
| P7 | - | 601 | - | .0448 | .0444 | .0434 | .0480 | 180 |

at the potential electrodes is composed of a message or useful signal having a frequency equal to that of the energizing current square wave, and random telluric noise, which we assume to have zero mean. Our purpose is to correctly evaluate the amplitude of the voltage signal so that accurate apparent resistivity can be computed. Eq. (13) was applied in several theoretical and experimental cases, and the results compared with those obtained using other spectral analysis techniques. Some of these results are summarized in Table 1. T1-T3 and NS1-NS7 are three theoretical and seven semi-theoretical processes, respectively. In each the message is given by a sine wave with frequency $f_s$=1/30 Hz and amplitude shown in column 1. With regard to the noise component, each theoretical process has pure white noise with a given power, while each semi-theoretical process includes one of the seven examples of telluric noise recorded in a geothermal area of Sardinia (Italy). Column 2 shows the number of data points (ND) for each recording uniformly sampled at 1 s intervals. Column 3 indicates the noise level, defined as the ratio of the standard deviation of the noise to the standard deviation of the signal. Columns 4, 5, 6 and 7 give the values of the signal amplitudes obtained with the MEM, MLM, periodogram and stacking methods. Finally, column 8 shows the length of the prediction error filter employed in the MEM, or the number of autocorrelation coefficients used in the MLM (e.g. Kanasewich, 1973).

For each process we studied the output from MEM and MLM by varying the number of coefficients (NC) from 15% to 60% of the number of data; the output amplitudes shown in Table 1 were computed for NC=30%ND. This percentage, at least for the cases taken into consideration, represents the minimum common value needed to obtain the output that we have empirically judged satisfactory. The correct choice of operator length is crucial (Chen and Stegen, 1974; Berryman, 1978); if the number is too small, the spectrum appears too smooth and featureless; if too large, the spectrum may show too much detail, amplifying the effect of the noise (unstable behaviour) and producing some spurious spectral peaks (splitting phenomena). In general we obtained the best results in the interval (25%, 45%).

The last three lines in Table 1 refer to three complete field recordings (P5-P7) relative to different spacings of two axial dipole soundings carried out for geothermal research in Sardinia (Italy). The energizing signal was a current square wave with frequency 1/30 Hz. Obviously, we do not have values in columns 1 and 3. It is worth noting that in this case all the estimates obtai-

**Table 3 -** Means of the percentage errors in the amplitude estimate.
1 MEM; 2 MLM; 3 Periodogram; 4 Stacking; T No trend removed; D After trend removal.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| T | 8.98 | 25.34 | 26.76 | 29.92 |
| D | 6.82 | 8.70 | 7.34 | 14.58 |

ned using the different methods, except for those given in column 7, refer to the amplitude of the fundamental harmonic of the square wave entering the recording equipment; to obtain the correct value of the voltage signal, the amplitude of the fundamental harmonic must be reduced, according to the Fourier analysis, by a factor $\pi/4$.

To give a visual representation of the processed data, records T1, NS1 and P7 are shown in Figs. 1 to 3 respectively. Figs. 4 to 6 show the corresponding maximum entropy power spectra. In these last figures the peak values are practically centred at the working frequency. We marked the 2 dB bandwidth of the peak whose value was used in eq. (13) to obtain the values of amplitudes shown in Table 1, column 4.

We stress the important practical implication of generalizing the solution to an $n$ dB bandwidth. Our procedure allows us to calculate the width of the peak (eq. (10)), and thus to satisfy eq. (13), even when the maximum entropy power spectrum does not have a spiked peak at the working frequency, or does not show one of the two 3 dB points. In this situation it is sufficient to do the computations giving $n$ a value smaller than 3 (we checked the validity of our approach with values of $n$ down to 0.05).

Fig. 7 shows the maximum entropy power spectrum of a semi-theoretical process similar to those discussed above; the peak does not have the 3 dB left point but it shows, for example, a 2 dB bandwidth.

Finally, a low frequency trend, probably due to chemical/physical processes occuring in the ground, visibly affects the record shown in Fig. 3. As this phenomenon occurs rather frequently, it is interesting to examine its possible influence on the determination of the signal amplitude.

A visual analysis of the processed records underlined the presence of a generally linear trend in the examples of telluric noise NS3-NS7 and in the complete field recordings P5-P7. After its removal using an easy procedure (Lapenna et al., 1987), the data were again processed and the results are displayed in Table 2: the estimates have visibly changed when using the MLM, periodogram and stacking, while they appear practically the same when MEM is applied.

## 4. Conclusions

A simple mathematical relationship between the maximum entropy spectral estimator and the amplitude of a sinusoid has been obtained. It has been applied to very noisy voltage recordings to evaluate correctly the amplitude of the useful voltage signal. The results displayed in Tables 1 and 2, together with those obtained with other spectral techniques, show that:

1) for recordings having pure white noise, MEM and MLM give a very good estimate of the

amplitude; the output from stacking is rather unsatisfactory (to obtain a better result, the number of data points was greatly increased; for instance up to 1321 for process T2 to have 0.0440 as output value);

2) the output from MEM is very good when examining the semi-theoretical processes; nevertheless, cases in which the output from the other methods are slightly better are also observed;

3) by comparing the results displayed in Tables 1 and 2 we see that the presence of low frequency trends does not greatly affect the evaluation of the signal amplitude, if MEM is applied; on the contrary, outputs from MLM, periodogram and stacking notably improve when a detrending processing is done on the records.

Table 3 shows the mean values of the percentage errors obtained when the four different methods are applied to processes NS3-NS7; line T is without trend removal, line D is after trend removal. The superiority of MEM is also evident.

In conclusion, we can state that for a given number of data points, noise level, and number of coefficients, MEM always shows a satisfactory estimate of the message amplitude. In most cases it gives the best results and no detrending processing is required. On the contrary, the other techniques applied sometimes show inconsistent results; in some cases they are very good, in others they are unsatisfactory, and a detrending processing is recommended.

Finally, we provide the reader with data on the computing time: using an IBM microcomputer (a 6150/25 machine), periodogram and stacking were the fastest techniques, even for long records (more than 1000 data points); on the contrary, MLM required the longest computing time (in this case the matrix of the autocorrelation coefficients must be inverted): about 17 minutes of CPU time were necessary to process a recording of 901 data points with a filter of 270 coefficients. For MEM, only 110 seconds of CPU were necessary to process the same data.

**Acknowledgement**. This work was supported financially by MURST (40% 92).

## References

Alfano L., Carrara E., Pascale A., Rapolla A. and Roberti N.; 1982: *Analysis procedure and equipment for deep geoelectrical soundings in noisy areas.* Geothermics, **11**, 269-280.

Berryman J. G.; 1978: *Choice of operator length for Maximum Entropy Spectral Analysis.* Geophysics, **43**, 1384-1391.

Burg J. P.; 1967: *Maximum entropy spectral analysis.* Presented at the 37th Ann. Internat. Mtg., Soc. Expl. Geophys.

Chen W. Y. and Stegen G. R.; 1974: *Experiments with maximum entropy power spectra of sinusoids*. J. Geophys. Res., **79**, 3019-3022.

Ciminale M. and Patella D.; 1982: *On the determination of the maximum likelihood power of very noisy voltage records in geoelectrical exploration.* Boll. Geof. Teor. Appl., **24**, 205-210.

Cuomo V., Lapenna V. A., Macchiato M., Patella D., Satriano C. and Serio C.; 1990: *Statistical analysis of noisy voltage recordings in geoelectrical prospecting.* Boll. Geof. Teor. Appl., **32**, 286-297.

Johnsen S. J. and Andersen N.; 1978: *On power estimation in maximum entropy spectral analysis.* Geophysics, **43**, 681-690.

Kanasewich E. R.; 1973: *Time sequences analysis in geophysics*. The University of Alberta Press, Edmonton.

Lacoss R. T.; 1971: *Data adaptive spectral analysis methods*. Geophysics, **36**, 661-675.

Lapenna V. A., Satriano C. and Patella D.; 1987: *On the methods of evaluation of apparent resistivity under conditions of low message-to-noise ratio.* Geothermics, **16**, 487-504.

Loddo M. and Patella D.; 1978: *Spectral analysis of voltage measurements for the interpretation of dipole geoelectrical soundings in low resistivity geothermal areas.* Geothermics, **6**, 227-235.

Radoski H. R., Fougere P. F. and Zawalick E. J.; 1975: *A comparison of power spectral estimates and applications of the maximum entropy method.* J. Geophys. Res., **80**, 619-625.

Satorius E. H. and Zeidler J. R.; 1978: *Maximum entropy spectral analysis of multiple sinusoids in noise.* Geophysics, **43**, 1111-1118.

Swingler D. N.; 1979: *A comparison between Burg's maximum entropy method and a nonrecursive technique for the spectral analysis of deterministic signals.* J. Geophys. Res., **84**, 679-685.

Ulrych T. J.; 1972: *Maximum entropy power spectrum of truncated sinusoids*. J. Geophys. Res., **77**, 1396-1400.

Ulrych T. J. and Bishop T. N.; 1975: *Maximum entropy spectral analysis and autoregressive decomposition.* Rev. Geophys. Space Phys., **13**, 183-200.