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# STUDY OF A MATHEMATICAL MODEL WITH CYLINDRIC SYMMETRY FOR CONVECTION IN THE UPPER LAYERS OF THE MANTLE

**Abstract.** Two-dimensional numerical models of steady state convection show that convection cells of large aspect ratio are possible for variable viscosity convection in the upper mantle. Our model includes the effects of variable viscosity, viscous dissipation, heat flow through the bottom and the adiabatic gradient. The large aspect ratio of the convection cells is primarily due to the large viscosity contrast between the lithosphere and the asthenosphere.

## INTRODUCTION

Convective motions in the mantle are possible through the quasi-fluid behaviour, on a geological time scale, of rocks with sufficiently high temperature and pressure; although for convective motions in the mantle, the usual velocity does not exceed 10 cm per year, such phenomenon exhibits qualitative characteristics typical of chaotic motion.

The mantle is a very special "fluid": it has a practically infinite Prandtl number ($Pr \approx 10^A$ 23) and a viscosity typically non-linear and dependent on temperature, pressure and state of tension. It is also compressible, with thermodynamic properties which depend on temperature and pressure, and melts changing its properties and redistributing its internal sources of heat.

Very detailed descriptions from a physical-mathematical point of view of thermal convection in general, and on convection in the mantle in particular, can be found in the quite recent scientific literature: some of the most recent articles are cited in the bibliography (Busse, 1978, 1981, 1989; Turcotte and Oxburgh, 1972, 1978; Richter, 1978; Schubert, 1979; Peltier, 1985, 1989; Hager and Gurnis, 1987; Roberts, 1987; Christensen, 1989; Jorvis and Peltier, 1989).

In this paper we analyze again a simple model, already described in studies of the "earth dynamo" (Busse, 1970, 1975, 1976), and we apply it to convective cells of the upper mantle, only for the part of the earth's surface bounded to the north by the Tropic of Cancer and to the south by the Tropic of Capricorn.

It would appear that variable viscosity greatly influences the convection pattern and that viscous dissipation may not always be negligible. For these reasons we have included these factors in the present model. Finally, the convection solutions are essential to obtain the temperature distribution within the upper mantle.



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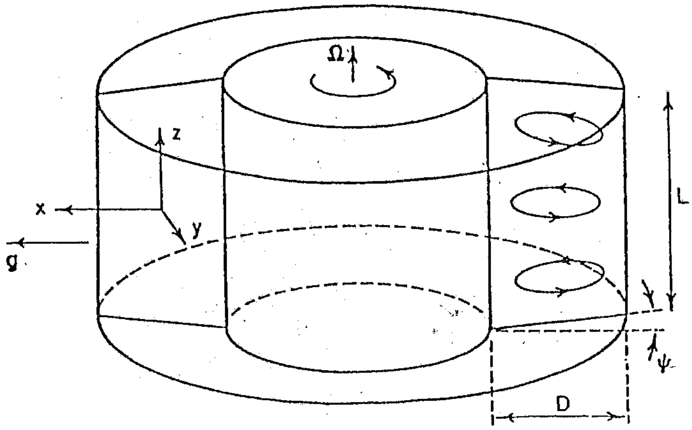


Fig. 1 — Geometry of the rotating annulus model with cylindric symmetry.

## GLOBAL ANALYSIS OF CONVECTION IN THE MANTLE

We consider the constraints on the convection of a fluid due to the rotation (see Fig. 1), driven by hydrostatic forces as given by (Boussinesq, 1903).

The motion is ruled by the momentum equation for viscous fluids and by the continuity equation, where the liquid is assumed to be incompressible (the variation of density appears only as a perturbation at the first order):

$$\frac{\partial \vec{u}}{\partial t} + (2\vec{\Omega} x \vec{u}) = -\frac{1}{\varrho_o} \vec{\nabla} P + \frac{\varrho'}{\varrho_o} \vec{g} + \nu \nabla^2 \vec{u} \qquad (1)$$

$$\vec{\nabla} \cdot \vec{u} = 0 \qquad (2)$$

In these equations, $\vec{\Omega}$ is the fluid angular velocity, $\vec{u}$ is the fluid velocity relative to the rotating frame, P is the pressure, $\vec{g}$ is the acceleration due to gravity plus centrifugal effects and $\nu = \eta/\varrho$ is the kinematic viscosity.

Here, the small variation in density $\varrho'$ is assumed to be maintained by the heating and/or cooling of the fluid. When the motion is slow and in a quasi-steady state, then the first and the last terms of eqn. (1) are negligible, and the curl of eqn. (1) leads to

$$(\vec{\Omega} \cdot \vec{\nabla})\, \vec{u} = 0 \qquad (3)$$

This is the theorem of Proudman-Taylor (PTT) (Proudman, 1916; Taylor, 1917, 1923, 1974), which says that the field is bidimensional and does not depend on the coordinate which is parallel to the rotation axis; so we can conclude that "slow steady motions relative to rotating axes must be two-dimensional" (Chandrasekhar, 1961).

## CONSEQUENCES OF THE PROUDMAN-TAYLOR THEOREM (PTT)

In those systems where the rotation cannot be neglected, the fluid tends to obey the PTT. If the nature of the main driving forces and the geometry of the fluid container do not allow the PTT to be satisfied, the motion is strongly restrained in some directions. These conditions can be clarified in the following example.

Let us consider a fluid which is confined between two horizontal planes separated by a

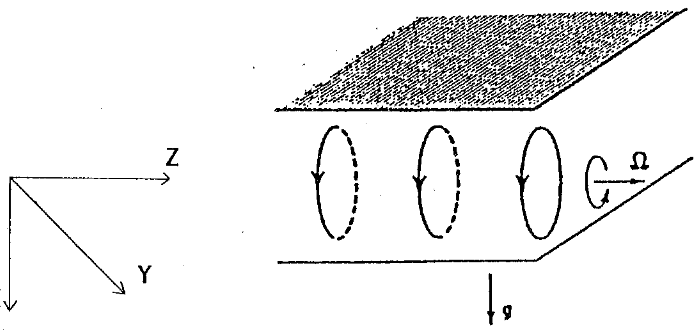


Fig. 2 — Preferred modes of convection in a Benard layer when $\vec{\Omega}$ is horizontal.

distance D. The fluid is heated from the bottom and cools at the surface, so that in the equilibrium state (absence of motion), the gradient of temperature $\beta$ is kept between the surface and the bottom.

When rotation with horizontal axis takes place, and instability is produced, characterized by motions independent of z and an axis parallel to the rotation axis (see Fig. 2).

This motion follows PTT and is influenced by the rotation only because it is coaxial with $\vec{\Omega}$ (there is no component of horizontal motion along z).

## THERMAL CONVECTION IN AN ISOVISCOUS LAYER OF THE MANTLE

If we consider a flat layer with free boundaries, at a fixed temperature, the solution for the convective motion can be described as follows (Loper, 1985).

Having defined the values of the state without motion with the subscript zero, we can define a stream function q as

$$q = kA \sin \frac{\pi x}{H} \sin \frac{\pi y}{\sqrt{2}H} , \tag{4}$$

and a temperature T as

$$T = To\ (x) - \frac{\sqrt{2}}{3\pi} \beta HA \sin \frac{\pi x}{H} \cos \frac{\pi y}{\sqrt{2}H} , \tag{5}$$

where $\beta$ is the gradient of temperature along x, H is the typical depth of the convective cell in question and k is the thermometric conductivity.

The thermal condition at each boundary depends on the heat transfer in the neighbouring medium. When it is very easy, either a fixed temperature or a fixed temperature gradient can be assumed. In this approximation, as a starting point, we are limiting ourselves to consider only the fundamental cell, although for a more realistic situation, many wave-numbers are needed (see e.g. (Stewart and Turcotte, 1989)).

From eqns. (1) and (2), the stress tensor for an incompressible fluid takes the form (Landau and Lifchitz, 1971)

$$\sigma_{ik} = -P\delta_{ik} + \nu\varrho\ (\partial V_i / \partial x_k + \partial V_k / \partial x_i ), \tag{6a}$$

$$\text{where } \vec{V} = (v,\ u)\ ;\ v = -\partial q / \partial y\ ,\ u = \partial q / \partial x,$$

and $\delta_{ij}$ is the Kronecker delta symbol.

If we consider only the x-direction (i=k=1), we have

$$\sigma_x = -P + 2\nu\varrho\ (\partial V_x / \partial x); \quad (6b)$$

that is

$$\sigma_x = -P + 2\nu\varrho\ (\partial v / \partial x), \quad (6c)$$

and therefore

$$\sigma_x = -P - 2\nu\varrho \frac{\partial^2 q}{\partial y \partial x} \ . \quad (7a)$$

From eqn. (4), $\partial^2 q / \partial x^2 = 2\ (\partial^2 q / \partial y^2)$, by the modified Navier-Stokes equations

$$\frac{dP}{dy} = \varrho\nu\ (\partial^2 u / \partial y^2 + \partial^2 u / \partial x^2). \quad (7b)$$

Integrating, P is obtained such that

$$P = \varrho\nu \left( \frac{\partial^2 q}{\partial y \partial x} + 2 \frac{\partial^2 q}{\partial y \partial x} \right) + Po\ (x), \quad (7c)$$

where Po (x) is the integration constant, which is clearly identified as a hydrostatic pressure by dimensional analysis. Remembering eqn. (4) and substituting in eqn. (7a), we have

$$\sigma_x = -Po\ (x) - \frac{5\ \pi^2}{\sqrt{2}} \frac{\nu\varrho k}{H^2} A \cos \frac{\pi x}{H} \cos \frac{\pi y}{\sqrt{2}H} \ . \quad (8a)$$

The perturbed boundaries are denoted by

$$x = \zeta_b\ (y); \qquad and \qquad x = H + \zeta_s\ (y). \quad (8b)$$

We shall distinguish between two kinds of bounding surfaces: rigid surfaces on which no slip occurs and free surfaces on which no tangential stresses act.

Considering first a rigid surface, the condition that no slip occurs on this surface implies that not only v, but also the horizontal component of velocity u, vanishes. Thus u=0 in addition to v=0 on a rigid surface.

Considering now a free surface, the condition that no tangential stresses act implies that P=0.

At the surface, the pressure is $P = Po\ (H) - \varrho_o\ g\ (H-x)$, where $\varrho_o = 0$ is the density over this surface; so Po (H)=0 and $\sigma_x\ (H) = -\varrho\ g\zeta_x$ of the Rayleigh number is given by $Ra = Ra_c = 27\ \pi^4 / 4$ (Chandrasekhar, 1961), i.e. $\nu / g = (\beta\ H^4\ \alpha) / Ra_c$, it follows that

$$\zeta_s = + \frac{10\sqrt{2}}{27\pi^2} \alpha\beta H^2 A \cos \frac{\pi y}{\sqrt{2}H} \ . \quad (8c)$$

Let us suppose that on the bottom, a fluid with density $\varrho_1$ is pushing, and the pressure, assumed to be hydrostatic, is $\varrho gH = Po\ (0) - \varrho_1\ gx$.

At level $\zeta_b$, this pressure equals $-\sigma_x\ (\zeta_b)$. Therefore

$$-\sigma_x\ (0) - \varrho g\zeta_b = Po\ (0) - \varrho_1\ g\zeta_b \ . \quad (9a)$$

From eqn. (8a), the values of $(\sigma_x + Po)$ at x=0 and at x=H are opposite:

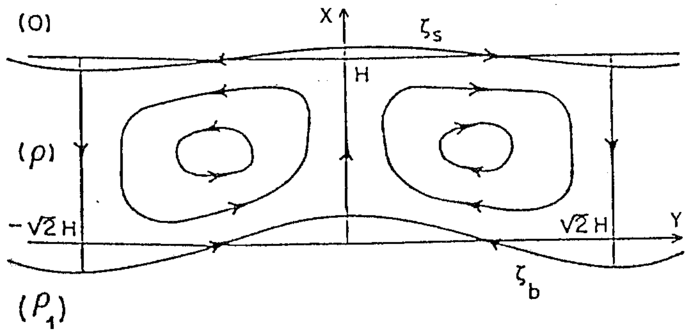


Fig. 3 — Two-dimensional marginal convection with stress-free boundaries.

$$\sigma_x\ (0)+Po\ (0)=-\sigma_x\ (H)=\varrho g\zeta_s\ . \tag{9b}$$

It follows that

$$\zeta_b=\frac{\sigma_x\ (0)+Po\ (0)}{(\varrho_1-\varrho)\ g}=\frac{\varrho}{\varrho_1-\varrho}\ \zeta_s\ , \tag{9c}$$

where $\varrho$ and $\varrho_1$ are the density over and under the bottom of the cell.

This convective motion has been drawn in Fig. 3, with the aid of eqns. (8) and (9c).

Thus, the dimensions of the cell along the y-axis are twice those along the x-axis. We had calculated previously that H is 700 km, including the lithosphere (thickness = 100 km); it comes out that the above described cell can be approximated by a rectangle whose sides are 700 and 2000 km.

If we list in the Table the parameters characterizing the convective cell for the three fundamental sets of conditions at the boundaries, we conclude that the greatest convective cell length is about 3000 km.

**Table — Convective cell parameters.**

| Nature of upper and lower surfaces | $R_c$ | a=k·H k=wave length | $\frac{2\pi}{a}$ | $l=2\sqrt{2}\cdot H$ cell length |
|---|---|---|---|---|
| Both free | 657.511 | 2.2214 | 2.828 | ≈2000 km |
| Both rigid | 1707.762 | 3.117 | 2.016 | ≈2806 km |
| One free and one rigid | 1100.65 | 2.682 | 2.342 | ≈2415 km |

## The equations

We consider therefore a horizontal infinite layer of fluid which rotates at a constant velocity called $\vec{\Omega}$. For the convective problem we are considering, it is enough to work with the equations of motion and the conduction of heat in the Boussinesq approximation. The only factors to be added, for the moment, are the effects of the Coriolis and centrifugal accelerations in the equations of motion.

Thus we have

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial}{\partial x_i}\left(\frac{P}{\varrho_o}-\frac{1}{2}|\vec{\Omega}\wedge\vec{r}|^2\right)+\left(1+\frac{\partial\varrho}{\varrho_o}\right)g_i+\nu\nabla^2u_i+ \tag{10}$$
$$+2\epsilon_{ijk}\,u_j\,\Omega_k\ ,$$

where $\epsilon_{ijk}$ is the total antisymmetric tensor.

Let us consider an initial state where a temperature gradient $\beta$ is kept constant and there is no motion. The equations ruling small perturbances to the first order give

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\varrho_o}\right) + g\alpha\vartheta i_i + \nu \nabla^2 u_i + 2\epsilon_{ijk} u_j \Omega_k , \tag{11}$$

where $\vec{i}=(1, 0, 0)$ is a unit vector in the direction of the x-axis, $\alpha$ is the coefficient of volume expansion and $\vartheta$ is the perturbation in temperature. Thus we have

$$\frac{\partial \vartheta}{\partial t} = \beta i_j u_j + k \nabla^2 \vartheta, \tag{12}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{13}$$

We can get rid of the term in $P/\varrho_o$ by using the curl from eqn. (11); then, knowing that

$$\epsilon_{ijk}\frac{\partial}{\partial x_i}\epsilon_{klm} u_l \Omega_m = \frac{\partial}{\partial x_i}(u_j \Omega_i - u_i \Omega_j) = \Omega_j \frac{\partial u_i}{\partial x_j} , \tag{14}$$

we get

$$\frac{\partial w_i}{\partial t} = g\alpha\epsilon_{ijk}\frac{\partial \vartheta}{\partial x_j} i_k + \nu \nabla^2 w_i + 2\Omega_j \frac{\partial u_i}{\partial x_j} . \tag{15}$$

Using the curl from this equation, we get

$$\frac{\partial}{\partial t}(\nabla^2 u_i) = g\alpha\left(i_i \nabla^2\vartheta - i_j\frac{\partial^2\vartheta}{\partial x_i \partial x_j}\right) + \vartheta \nabla^4 u_i - 2\Omega_j \frac{\partial w_i}{\partial x_j} . \tag{16}$$

Finally, if we multiply eqns. (15) and (16) by $\vec{i}$, we obtain the general equations

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta + 2\Omega_j \frac{\partial v}{\partial x_j} , \tag{17}$$

$$\frac{\partial}{\partial t} \nabla^2 v = g\alpha\left(\frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2}\right) + \nu \nabla^4 v - 2\Omega_j \frac{\partial \zeta}{\partial x_j} . \tag{18}$$

Now assuming that $\vec{\Omega}$ has the same direction as the z-axis, and remembering the hypothesis on $\vec{g}$ that

$$\vec{g} = g\ (1, 0, 0), \tag{19}$$

$$\vec{\Omega} = \Omega\ (0, 0, 1) \tag{20}$$

then eqn. (17) becomes

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta + 2\Omega \frac{\partial v}{\partial z} , \tag{21}$$

where $\zeta$ and v are the x-components (vertical components) of vorticity and velocity respectively.

Since $\zeta$ is a component of vorticity ($\simeq$ curl $\vec{V}$), it has dimensions $s^{-1}$; so that $\left[\frac{\partial \zeta}{\partial t}\right] = s^{-2}$.

We now transform eqn. (21) into an equation where all the quantities are dimensionless.

We indicate with L a characteristic quantity (linear dimension of a real cell), and make the following changes:

$$\tau = \frac{\nu t}{L^2}\ ;\ X = \frac{x}{L}\ ;\ Y = \frac{y}{L};\ Z = \frac{z}{L}\ ;\ \Theta = \frac{vL}{\nu}\ ;\ \Gamma = \frac{\Omega L^2}{\nu}\ ;\ \Delta = \frac{\zeta L^2}{\nu}$$

(slowed times, reduced quantities).

It is straight forward to see that all the quantities are dimensionless. By a simple substitution we get

$$\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial Z^2} + 2\Gamma \frac{\partial \Theta}{\partial Z}\ . \tag{22}$$

The dimension of the coefficients of the right-hand side is of the order of $\frac{1}{2\Gamma} \approx \frac{10^{21}}{L^2} \approx 10^9$ (characteristic length $L \approx 10^6$).

On the whole, the last term on the right-hand side is negligible. This result leads to a diffusion equation with solutions for $\zeta$ which are particularly stable in time and with high symmetry.

An alternative idea is to look for a solution where v (x-component of velocity) is rapidly varying with respect to z. In this case $\zeta$ (x-component of vorticity) could be relatively stable in space and approximate an harmonic function.

Then $\nabla^2\Phi$ could be objectively small with respect to $(\vec{\Gamma} \cdot \vec{\nabla})\,\Theta$, so balancing the large ratio between the coefficients: this would imply a certain asymmetry of v, when we move on the plane zy.

Such a situation, which is outside the PTT, should be analyzed further.

## CONCLUSIONS

The cylindric model considered seems an interesting model for the area between the two Tropics.

Convection is confined inside the external ring that rotates around its symmetry axis; it can be represented, approximately, by concentric circles with axis parallel to that of rotation, since the small slope of the upper and lower borders of the ring guarantees that the Proudman-Taylor-Theorem (PTT) is satisfied.

The results obtained for a model with constant viscosity are shown in Fig. 3; the Rayleigh number in the case considered is $10^6$.

The stream function is symmetric with respect to the average point of the rising flow and this symmetry is not significantly influenced by the Earth's rotation.

Finally it should be clear that the maximum of the stream function is situated in the descending area of the cell; as a consequence, the vertical velocity in the descending area is greater than in the ascending, and the maximum of its velocity is on the surface, approximately above the maximum of the stream function.

These simple, constant viscosity solutions probably have little relation to the convective states inside the Earth. For example, the temperatures rise too slowly to serve as a realistic representation of what occurs in the upper layers of the mantle.

The results obtained for a model with viscosity varying with depth seem more realistic;

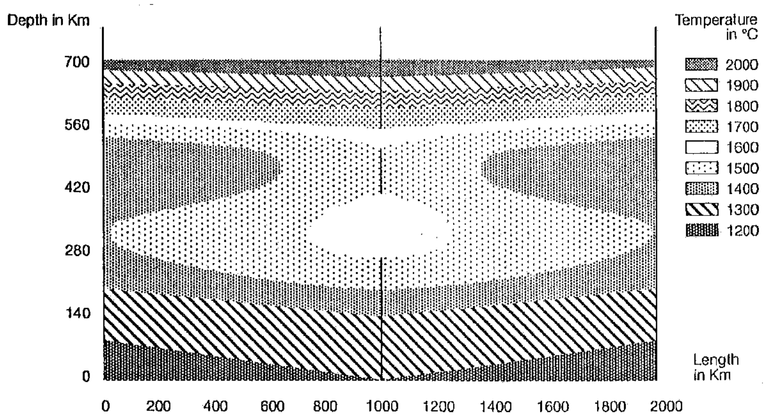


Fig. 4 — Convective cell temperatures in the variable viscosity case with bottom heat flux.

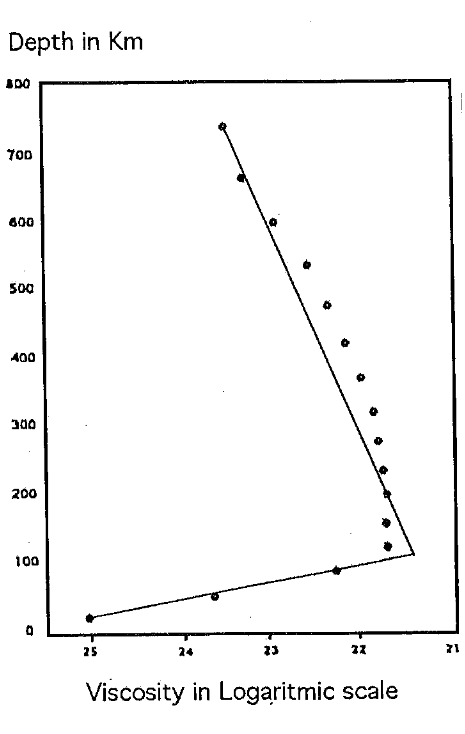


Fig. 5 — Assumed kinematic viscosity profile as a function of depth only.

in fact it can be noted in Fig. 4 that the model, driven by a flow of heat that rises from the bottom towards the top, shows a region over the bottom where the isotherms are nearer than those situated in the layer under the surface.

This result would seem to be reliable enough even considering the fact that viscosity was made to vary linearly with depth (see Fig. 5), and the angular point that appears at the boundary between the lithosphere and the astenosphere could be the cause of a break in the continuity of temperature in this zone. Better approximations, in the study phase, should resolve the problem.

Under the geodynamic profile the results attained in the study show three points of primary interest:

1) The theoretical developments of the model demonstrate that, even by varying greatly the conditions in the surroundings, the single convective cells cannot go much beyond 3000 kilometers in amplitude. This does not match the real dimensions of the plates, which should be drawn from the convective currents in real situations.

Also considering the fact that the descending portion of the cell is formed in part or completely by the cold lithosphere in subduction, at most one can say that the convective currents can be the active cause of plate movements around ridges but not over long distances.

2) The numerical variable viscosity model suggests that the lateral expansion of the warm rising material appears, however, at depths too great to influence the movement of the plates.

The active convection model as the only cause of movement (extreme conception) of the plates is rendered completely invalid by the previous considerations and the "softer" conception is also questioned. Further improvement of the study is necessary to verify these conclusions.

3) An interesting suggestion is offered by the numerical model of variable viscosity: it is represented by matter at a high temperature that one finds at a minimum depth of 200 kilometers, detached from the base of the asthenosphere.

Apart from the other considerations such an element could represent a preliminary (embrionic) model for the stoking of a hot spot of the "changeable" kind (not rooted in the lower mantle).

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