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E. E. S. SAMPAIO [1] and P. M. CARRION [2]

ELECTROMAGNETIC FIELDS ABOUT A FLAT CIRCULAR ELECTRODE

Abstract. Although different sources have been studied in geophysics, a rigorous study of the role of surface distributed sources of alternating current is virtually missing in the geophysical literature. In fact, sources of electromagnetic radiation defined at surfaces play an important role and can emerge in many practical applications. The goal of this paper is to solve the problem of electromagnetic radiation generated by an alternating current dipole distributed on a perfectly conducting thin circular disk, and perpendicular to it. So the importance of flat electrodes as a source of an electromagnetic field in geophysical exploration can be understood. The problem entails a solution of linear Maxwell's equations in the frequency domain, using the Hankel transform applied to modified Helmholtz wave equations. This procedure reduces the wave equations to a system of ordinary differential equations with respect to the vertical coordinate. Unknown coefficients are sought from the continuity of boundary values for the field components and jump discontinuities of their first derivatives at the source location. Different complexities such as underlying multilayered media can be easily compounded to a solution by increasing the number of equations satisfying the boundary conditions at the interfaces. The solution obtained allows calculation of both the primary and the scattered electromagnetic field components radiated from disk sources of alternating current situated in a conductive media. The computed values of the azimuthal component of the magnetic field, and the radial component of the electric field show that for distances less than one hundred disk radii the contribution of a flat electrode is significant. Because of this and the finite size and geometry of the disk source it is advantageous to employ it for parametric sounding with a small distance between transmitter and receiver. For larger distances the flat electrode may be adequately employed so that the magnitude of the produced field is much lower than the magnitude of the anomaly field induced by the shielded cables.

INTRODUCTION

A fundamental problem in exploration geophysics is to establish the relationship between the field scattered by the earth and its sources. For electromagnetic methods the propagation of waves excited by loop sources or cables carrying alternating current is well understood. Wait (1982) as well as Ward and Hohmann (1987) give a comprehensive review of the subject. Besides these sources, several electromagnetic exploration techniques, such as Controled Source Audio-Magnetotelluric (CSAMT), Spectral Induced Polarization (SIP), Long-Offset Transient Electromagnetic (LOTEM), TURAM and Well-Logs employ electrodes at both ends of shielded cables carrying alternating current. The electrodes act as additional sources of the electromagnetic energy, besides the fields induced by the cables, and induced polarization effects.

In SIP surveys the role of the electrodes is predominant, because the electromagnetic coupling effects between the transmitting and receiving cables are corrected so as not to mask the induced polarization response (Zonge and Hughes, 1981). Therefore it is necessary to know the induced field from the electrodes to separate it from the contribution due to the induced polarization

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[1] PPPG/UFBA, Instituto de Geociências, Campus Universitário da Federaçao, 40210 Salvador, Bahia, Brasil.

[2] Osservatorio Geofisico Sperimentale, P.O. Box 2011, 34016 Trieste, Italy.

effect. On the other hand, in TURAM type surveys (Parasnis, 1966) the shielded cable plays the predominant role. Therefore it is necessary to remove the contribution of the electrodes so that the anomaly induced by the cable can be properly understood. This is specially true if the survey is close to shallow lateral heterogeneities. A similar reasoning can be applied to Well-Log surveys. For CSAMT and LOTEM (Strack et al., 1990) surveys it may be necessary to perform the same type of correction, depending upon the relative magnitude of the field radiated from the electrodes, and the anomaly field induced by the cable at the receiver site. Whether a source of signal or a source of noise, it is useful to to compute the electromagnetic field generated by the electrodes, so that the field data can be interpreted appropriately.

Among various shapes, the electrodes can be flat, such as the button electrodes described by Dakhnov (1962) in terms of their contact resistance for well-logs, and the plate electrodes described by Sunde (1968) related to resistance of grounding arrangements. The flat circular electrode has a large contact area compared to a stake electrode driven into the ground, and has a small thickness. It also does not create substantial perturbation to the field components close to it due to the source geometry and size, or due to mathematical singularity problems (Jackson, 1975). Therefore it normally has a reduced contact resistance, it presents no restriction to measurements close to it, and it is amenable to modelling as a disk source.

The goal of this paper is to give an explicit treatment of a circular disk source of alternating current, so the importance of flat electrodes as a source of an electromagnetic field can be understood. It will determine an analytical solution of the wave field components, both magnetic and electric, generated by a disk-type source in the frequency domain. For the sake of simplicity we consider two homogeneous conductive half-spaces separated by a horizontal surface, and a horizontal circular disk source of vertical electric current placed in the lower half-space and with the center at the origin of a cylindrical coordinate system (see Fig. 1). Solving the problem with respect to Hankel transformed wave fields, physical components of the time-harmonic wave fields are then found by inverse Hankel transform. Such wave fields are subsequently computed for different observational points in space. A time dependence of the form $exp\ (i\omega t)$ is considered thouroughout.

MODEL AND BASIC EQUATIONS

The electric field $\vec{E}$ and the magnetic field $\vec{H}$ are coupled through Maxwell's equations:

$$\nabla \times \vec{E}\ (\vec{x},\omega) = -j\omega\mu\vec{H}\ (\vec{x},\omega), \tag{1}$$

and

$$\nabla \times \vec{H}\ (\vec{x},\omega) = \sigma\ (\vec{x},\omega)\ \vec{E}\ (\vec{x},\omega) + \vec{j}_z, \tag{2}$$

where $\vec{x} \in R^3$ is a coordinate vector in 3-D Euclidean space, μ is the magnetic permeability, ω is the angular frequency, σ stands for the total complex conductivity, and j_z denotes the source function. The source function for this problem consists of an alternating vertical electric dipole distributed on a perfectly conducting circular thin disk of radiusa. Following the procedure described in Stratton (1941) as a limiting case for an oblate spheroid, the source functions for this disk-type source can be written in cylindrical coordinates $(r,\ \phi,\ z)$ as

$$j_z = \frac{I \quad h\ (a-r)}{2\pi a \sqrt{a^2-r^2}}, \qquad for\ z=0, \tag{3}$$

where $h\ (a-r)$ is the Heavside function, and I is the total current flowing across the disk. Eqn. (3) is valid for the static case, and is approximately valid for an alternating current, as long as the radius a is much less than a wavelength. For an electric dipole $I\Delta z$, and Δz small, eqn. (3) can be rewritten to define the approximate value of the source function at any point $(r,\ \phi,\ z)$ as

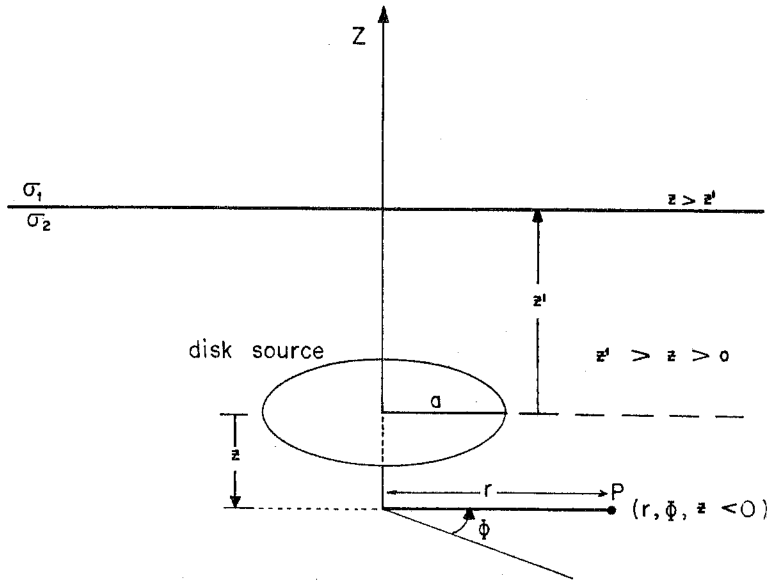

Fig. 1 - A horizontal circular disk source placed below the interface between two media with different conductivities.

$$j_z=\frac{I\Delta z \;\; h\,(a-r)}{2\pi a\sqrt{a^2-r^2}}\,\delta\,(z). \tag{4}$$

Let two homogeneous and isotropic half-spaces, with the horizontal interface at $z=z'$, have conductivities σ_1 and σ_2 respectively (Fig. 1). A thin circular perfectly conductive disk electrode is placed in the second medium at $z=0$. The axial symmetry of the problem leads to an invariance with the coordinate ϕ, and to a magnetic field H that has only the ϕ component: $H=(0, H, 0)^T$. Maxwell's eqns. (1) and (2) can then be reduced to the following modified Helmholtz wave equations with respect to the azimuthal component H:

$$[\nabla^2-\frac{1}{r^2}+k_1^2]\,H\,(r,\,z,\,\omega)=0, \qquad if\ z\text{»}z' \tag{5}$$

and

$$[\nabla^2-\frac{1}{r^2}+k_2^2]\,H\,(r,\,z,\,\omega)=-\frac{\partial}{\partial r}j_z, \qquad if\ z\text{»}z' \tag{6}$$

In eqns. (5) and (6), $k_{1,\,2}$ represents the complex wave-number defined as:

$$k_{1,\,2}^2=-i\omega\mu\sigma_{1,\,2}. \tag{7}$$

The right-hand side of eqn. (6) represents the source responsible for the field. Has a negative sign because the positive cicle of the vector j_z is directed in the negative sense of the z coordinate (see Fig. 1).

SOLUTION TO THE PROBLEM

Let us apply a Hankel transform to eqns. (5) and (6) in the following form:

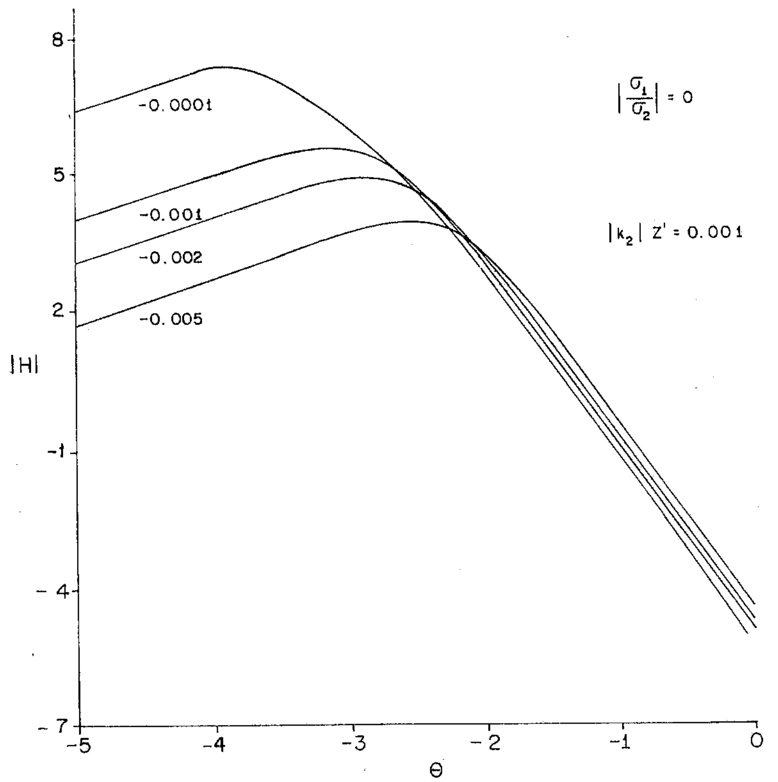

Fig. 2 - Logarithm of the azimuthal magnetic field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of the source from the interface $\bar{z}' = 0.001$ and the ratio of the conductivities $(\sigma_1/\sigma_2) = 0$.

$$\tilde{H}\ (m,\ z) = \int_0^\infty H\ (r,\ z)\ J_1\ (mr)\ rdr. \tag{8}$$

Its inverse is given by

$$H\ (r,\ z) = \int_0^\infty \tilde{H}\ (m,\ z)\ J_1\ (mr)\ mdm. \tag{9}$$

This yields:

$$\left[\frac{\partial^2}{\partial z^2} + (k_1^2 - m^2)\right] \tilde{H}\ (m,\ z) = 0 \qquad \text{for } z \gg z' \tag{10}$$

and

$$\left[\frac{\partial^2}{\partial z^2} + (k_2^2 - m^2)\right] \tilde{H}\ (m,\ z) = -\int_0^\infty \frac{\partial j_z}{\partial r} J_1\ (mr)\ rdr \qquad \text{for } z \ll z'. \tag{11}$$

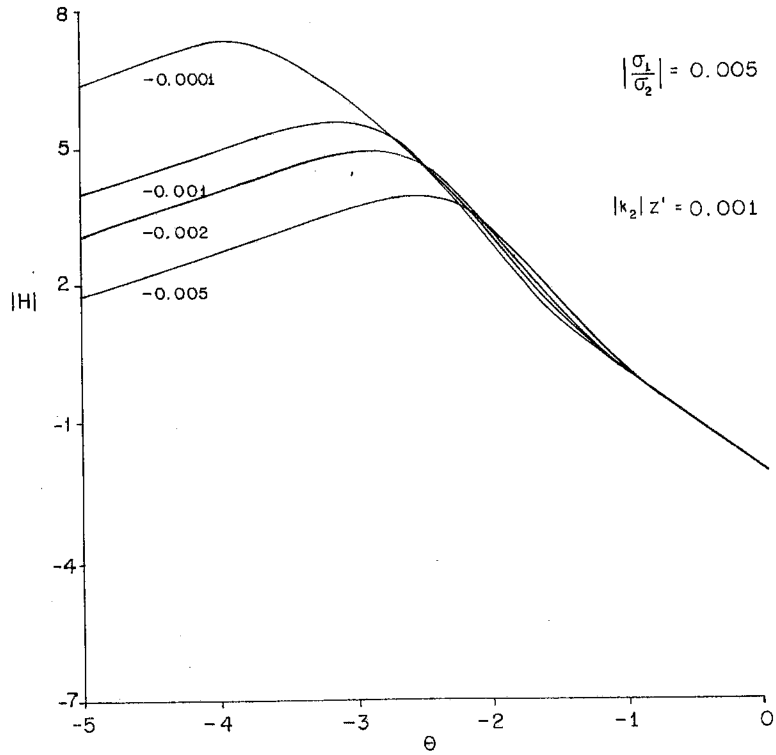

Fig. 3 - Logarithm of the azimuthal magnetic field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of the source from the interface $\bar{z}' = 0.001$ and the ratio of the conductivities $(\sigma_1/\sigma_2) = 0.005$

The right-hand side of eqn. (11) can be explicitly computed using integration by parts:

$$\int_0^\infty \frac{\partial j_z}{\partial r} [J_1 (mr)\, r]\, dr = -\int_0^\infty j_z \left[J_1 (mr) + \frac{\partial J_1 (mr)}{\partial r} r\right] dr. \tag{12}$$

Recalling the following identity for the Bessel functions (Mc Lachlan, 1955):

$$J_1 (mr) + \frac{\partial J_1 (mr)}{\partial r} r = mrJ_0 (mr), \tag{13}$$

integral (12) can be further simplified to

$$\int_0^\infty \frac{\partial j_z}{\partial r} J_1 (mr)\, rdr = -\int_0^\infty j_z mrJ_0 (mr)\, dr. \tag{14}$$

Using the following relation (Gradshteyn and Ryzhik, 1980)

$$\int_0^1 \frac{\chi J_0 (\chi y)}{\sqrt{1-\chi^2}} d\chi = \frac{\sin (y)}{y}, \tag{15}$$

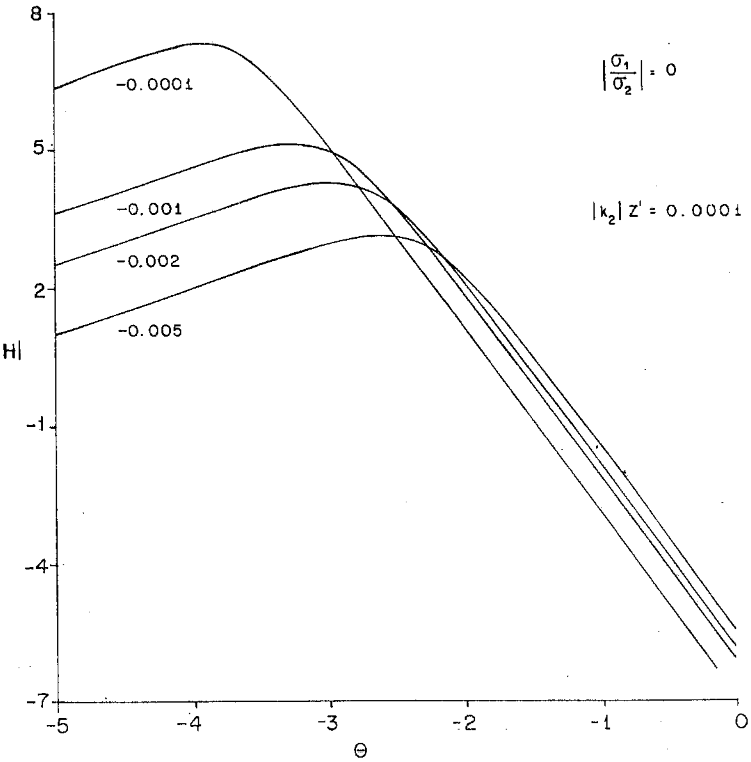

Fig. 4 - Logarithm of the azimuthal magnetic field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of the source from the interface $\bar{z}' = 0.0001$ and the ratio of the conductivities $(\sigma_1/\sigma_2)=0$.

then (14) with (4) reduces to

$$\int_0 \frac{\partial j_z}{\partial r} J_1\,(mr)\; rdr = -\frac{I\Delta z}{2\pi a} \sin\,(ma)\;\delta\,(z). \tag{16}$$

Iserting (16) into (11) yields an ordinary differential equation with a singular right hand side:

$$[\frac{\partial^2}{\partial z^2} + (k_2^2 - m^2)\,]\tilde{H}\,(m,\, z) = \frac{I\Delta z}{2\pi a} \sin\,(ma)\;\delta\,(z), \qquad if\; z \ll z'. \tag{17}$$

Solving eqns. (10) and (17) for $\tilde{H}\,(m, z)$, applying the boundary conditions (see Appendix) and taking the inverse Hankel transform of $\tilde{H}\,(m, z)$ yields the expression for the magnetic field component in the three regions:

$$H\,(r,\, z) = -\frac{I\Delta z}{2\pi\alpha} \int_0^{\infty} \sin\,(ma)\, \frac{c^{-\sqrt{(m^2-k^2_1)}\,(z-z') - \sqrt{(m^2-k^2_2)}\,z'}}{[\sqrt{m^2-k_2^2} + \left(\frac{\sigma_2}{\sigma_1}\right)\sqrt{m^2-k_1^2}]}\, mJ_1\,(mr)\; dm, \qquad z \geq z', \tag{18}$$

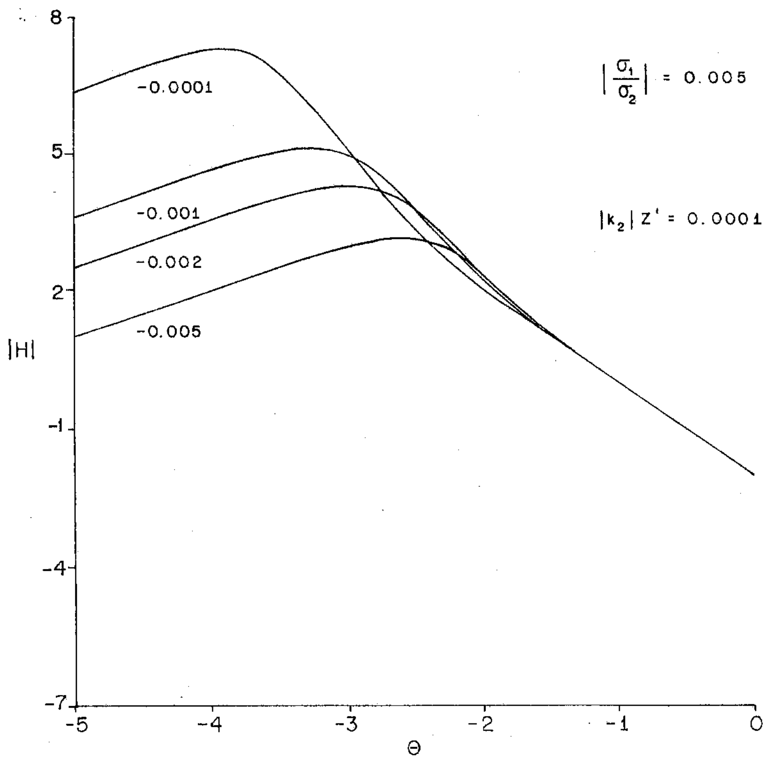

Fig. 5 - Logarithm of the azimuthal magnetic field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of source from the interface $z' = 0.0001$ and the ratio of the conductivities $(\sigma_1/\sigma_2) = 0.005$

$$H(r, z) = -\frac{I\Delta z}{4\pi a}\left[\int_0^{\infty} \frac{\sin(ma)}{\sqrt{m^2-k_2^2}} e^{-\sqrt{m^2-k_2^2}\,z}\, mJ_1(mr)\, dm + \int_0^{\infty} \frac{\sin(ma)}{\sqrt{m^2-k_2^2}} \times \right. \tag{19}$$

$$\left. \frac{e^{\sqrt{m^2-k_2^2}\,(z-2z')}\left[\sqrt{m^2-k_2^2}-\left(\frac{\sigma_2}{\sigma_1}\right)\sqrt{m^2-k_1^2}\right]}{\left[\sqrt{m^2-k_2^2}+\left(\frac{\sigma_2}{\sigma_1}\right)\sqrt{m^2-k_1^2}\right]} mJ_1(mr)\, dm\right], \quad z' \geqslant z \geqslant 0,$$

and

$$H(r, z) = -\frac{I\Delta z}{4\pi a}\left[\int_0 \frac{\sin(ma)}{\sqrt{m^2-k_2^2}} e^{\sqrt{m^2-k_2^2}\,z}\, mJ_1(mr)\, dm + \int_0 \frac{\sin(ma)}{\sqrt{m^2-k_2^2}} \times \right. \tag{20}$$

$$\left. \frac{e^{\sqrt{m^2-k_2^2}\,(z-2z')}\left[\sqrt{m^2-k_2^2}-\left(\frac{\sigma_2}{\sigma_1}\right)\sqrt{m^2-k_1^2}\right]}{\left[\sqrt{m^2-k_2^2}+\left(\frac{\sigma_2}{\sigma_1}\right)\sqrt{m^2-k_1^2}\right]} mJ_1(mr)\, dm\right], \quad z \leqslant 0,$$

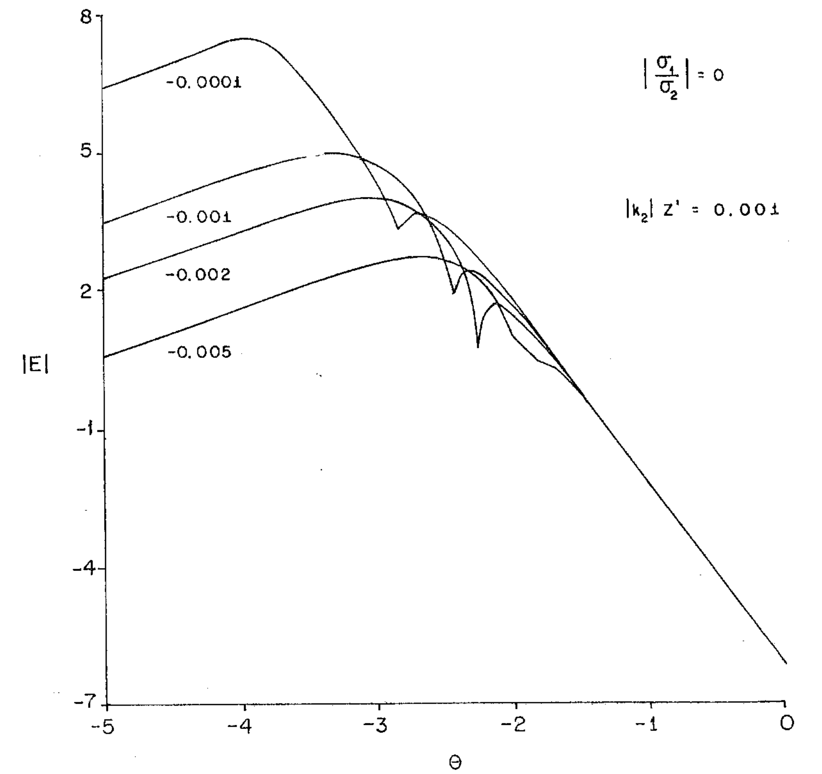

Fig. 6 - Logarithm of the radial electric field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of the source from the interface $z' = 0.001$ and the ratio of the conductivities $(\sigma_1/\sigma_2)=0$.

Consequently the radial comonent of electric field (E) has the following expression for $z \leqslant 0$:

$$E\ (r,\ z) = \frac{I\Delta z}{\pi\sigma_2 a}\left[\int_0 \sin\ (ma)\ c^{\sqrt{(m^2-k_2^2)}z}\ mJ_1\ (mr)\ dm + \right.$$

$$\left.\int_0^\infty \sin\ (ma)\ \frac{e^{\sqrt{(m^2-k_2^2)}(z-2z')}\ [\sqrt{m^2-k_2^2}-\left(\frac{\sigma_2}{\sigma_1}\right)\sqrt{m^2-k_1^2}]}{[\sqrt{m^2-k_2^2}+\left(\frac{\sigma_2}{\sigma_1}\right)\sqrt{m^2-k_1^2}]}\ mJ_1\ (mr)\ drm\right], \quad (21)$$

where Real $\left(\sqrt{m^2-k_{1,\,2}^2}\right) > 0.$

NUMERICAL RESULTS

Let us consider the following example: a half-space of conductivity σ_2 containing the disk source beneath air of conductivity σ_1. Introducing a new integration variable χ defined as $m=|k_2|\chi$, and neglecting second and higher-order terms of the ratio (σ_1 / σ_2), the integrals (20) and (21) reduce to

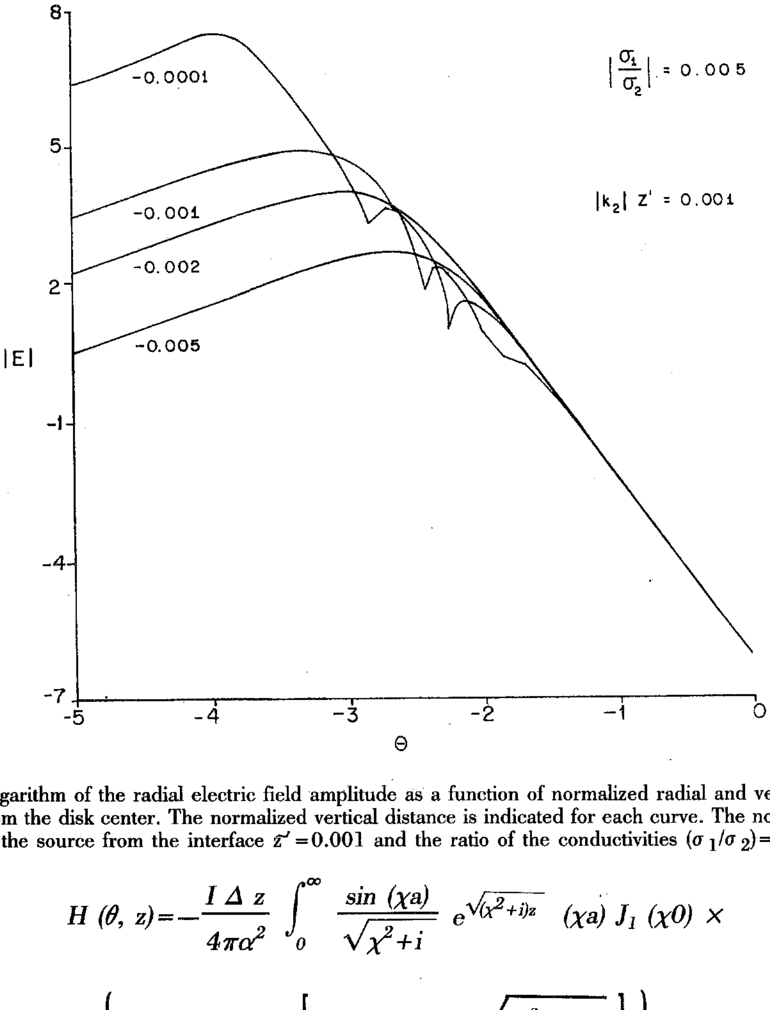

Fig. 7 - Logarithm of the radial electric field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of the source from the interface $z'=0.001$ and the ratio of the conductivities $(\sigma_1/\sigma_2)=0.005$.

$$H(\theta, z) = -\frac{I\,\Delta z}{4\pi a^2} \int_0^\infty \frac{\sin(\chi a)}{\sqrt{\chi^2+i}}\, e^{\sqrt{(\chi^2+i)}z}\, (\chi a)\, J_1(\chi 0) \times$$

$$\left\{ 1 - e^{-2\sqrt{(\chi^2+i)}z'} \left[1 - 2\left(\frac{\sigma_1}{\sigma_2}\right) \sqrt{\frac{(\chi^2+i)}{(\chi^2+\frac{\sigma_1}{\sigma_2}\, i)}} \right] \right\} d\chi, \qquad (22)$$

and

$$E(\theta, z) = \frac{I\Delta z}{4\pi\sigma_2\, a^3} \int_0^\infty \sin(\chi a)\, e^{\sqrt{(\chi^2+i)}z}\, (\chi a^2)\, J_1(\chi\Theta) \times$$

$$\left\{ 1 - e^{-2\sqrt{(\chi^2+i)}z'} \left[1 - 2\left(\frac{\sigma_1}{\sigma_2}\right) \sqrt{\frac{(\chi^2+i)}{(\chi^2+\frac{\sigma_1}{\sigma_2}\, i)}} \right] \right\} d\chi, \qquad (23)$$

where $a=a|k_2|$, $\theta=r|k_2|$, $z=z|k_2|$, and $z'=z'|k_2|$ are dimensionless parameters. We set $10^{-5} \leq 0 \leq 1$; $|\sigma_1 / \sigma_2|=0$ and 5×10^{-3}; $a=10^{-4}$; $z'=10^{-4}$ and 10^{-3}; and $z=-10^{-4}$, -10^{-3}, -2×10^{-3} and -5×10^{-3}. The total current I was chosen such that

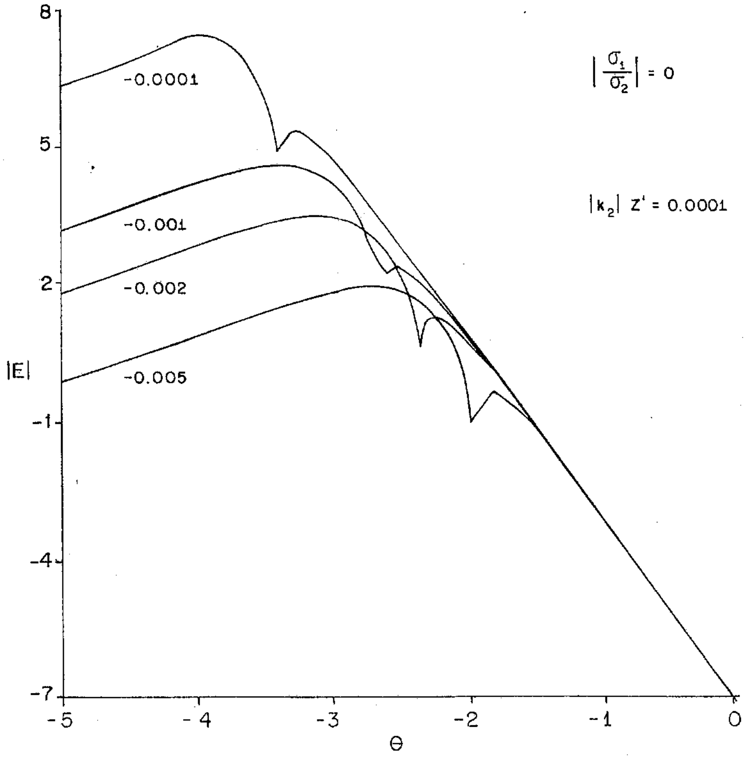

Fig. 8 - Logarithm of the radial electric field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of source from the interface z' =0.0001 and the ratio of the conductivities $(\sigma_1/\sigma_2)=0$.

$$\frac{I\Delta z}{4\pi \alpha^2} = 10^8 \ (nT),$$

and

$$\frac{I\Delta z}{4\pi\sigma_2 \ \alpha^3} = 10^8 \ (mV/km).$$

Figs. 2-5 represent the variation of the absolute value of the magnetic field as a function of 0 on a log-log scale for four different values of z. All the curves reach a maximum for $\theta_{max} > \alpha$, θ_{max} increasing with z, and converge to zero as $0 \rightarrow \infty$. For each value of z and z', and for $0 < 10^{-2}$ the curves are identical in shape and in magnitude, but they decrease in magnitude with decreasing z'. For each value of z and for $0 > 10^{-2}$ the shape of the curves is identical for the same value of $(\sigma_1 \ / \ \sigma_2)$, and the magnitude decreases with decreasing z'. Figures 6-9 represent the variation of the absolute value of the radial electric field as a function of θ on a log-log scale for four different value of z. The general behaviour of the electric field curves is identical to the magnetic field curves with one important exception: all the curves pass through a minimum for $\theta_{min} > 10^{-3}$, θ_{min} increasing with z, caused by a large phase shift between the primary and secondary fields.

DISCUSSION ON APPLICATION

In actual field surveys the primary plus induced electromagnetic fields contributed by the

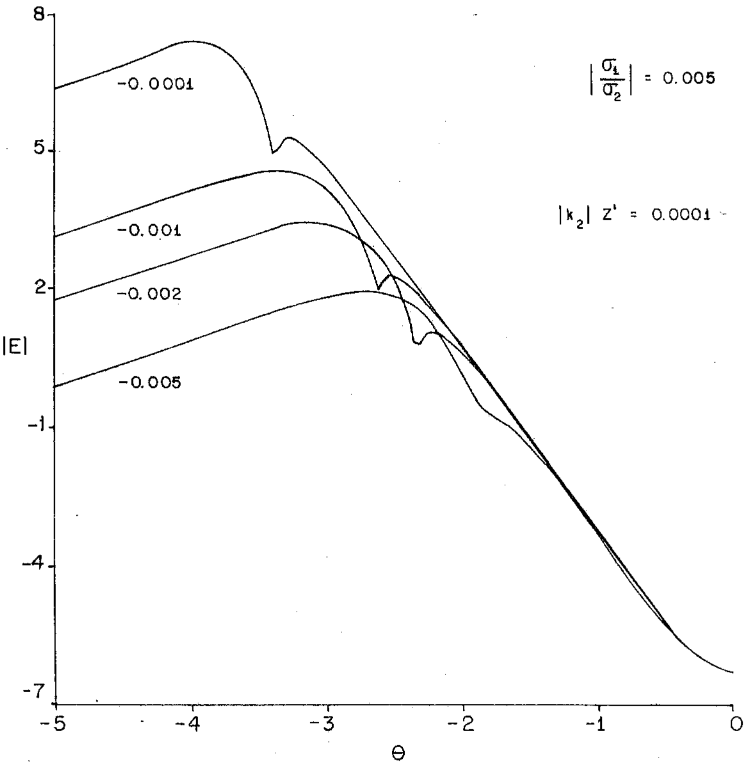

Fig. 9 - Logarithm of the radial electric field amplitude as a function of normalized radial and vertical distances from the disk center. The normalized vertical distance is indicated for each curve. The normalized depth of the source from the interface $z' = 0.0001$ and the ratio of the conductivities $(\sigma_1/\sigma_2) = 0.005$.

electrodes, the shielded cable, and, if present, the induced polarization effect add to a total field measurement at the receiver location. The situation becomes more difficult because of the existence of noise, lateral heterogeneity, and anisotropy. Therefore, an appropriate analysis of a survey result also requires identification of the fields produced by each source from the raw data. Our purpose is to show the contribution of flat electrodes, which we consider to be more advantageous than stake electrodes for the following reasons: (1) they are simpler to deal with mathematically; and (2) the flow of primary current presents a single direction perpendicular to the disk.

Figs. 2-9 have been computed assuming $\sigma_2 = 0.01$ (siemens/m), and $\omega = \frac{10^3}{4\pi}$ (rd/s), if we consider $I = \frac{4\pi}{10}$ (ampere), $\Delta z = 0.01$ (m) and $a = 0.1$ (m). For these values of a flat electrode of alternating current in a conductive half-space, we observe that $r = 10$ (m) is the boundary between "signal" and "noise" of the radiated field components. So for $r \leq 100$ a both the radial component of the electric field and the azimuthal component of the magnetic field have sufficient magnitude to be employed in the investigation of the half-space. This will certainly be the case of electromagnetic well-logs, and electromagnetic parametric sounding employing four electrodes in an L-shaped array. Also the magnitude and the variation with frequency of the electric field generated by flat electrodes carrying alternating current should be employed to analyze induced polarization data after removal of the coupling effect from the cables, instead of the well known formula $\varrho_a = \frac{V}{I} \times K$ which is valid only for a direct current point source.

Since for r > 100 a the field decarys appreciably, flat electrodes can be employed in situations where it is necessary to neglect its contribution. However, care must be taken to ensure that the field from the electrodes is not only much smaller than the field radiated by the cable, but also negligible compared to the anomaly field under investigation. Eqns. (22) and (23) show that the ideal condition happens when both σ_1 and z' can be neglected.

CONCLUSION

In this paper, we considered the radiation pattern of a disk type source located below and at the earth surface. An explicit relation in closed form was obtained for the magnetic and electric field components. We considered a simplified model where the disk source is located in a homogeneous half-space. This was done to obtain a clear picture of the radiation pattern of the disk source. Different complexities such as layered media can be compounded using a similar procedure as described in the paper.

The results obtained show that it is wrong to neglect the role of the electrodes in electromagnetic surveys, except if all the following conditions occur: (1) electrode size is much smaller than transmitter-receiver distance; (2) conductive scatterers are far from the electrodes; and (3) there is negligible induced polarization effect. At least in well-log surveys, as well as in ground surveys for disseminated sulphide exploration, these conditions are not all met. The results can also be employed to improve the execution and the evaluation of surveys using shielded cables grounded by flat electrodes.

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APPENDIX

Let us divide the whole space into three subspaces. In the first subspace z > z′ a solution can be uniquely presented as

$$H\ (m,\ z)=A\ (m)\ e^{-\sqrt{(m^2-k^2{}_1)}z}, \qquad for\ z>z'. \tag{A-1}$$

In the second subspace (z′ > z > 0) a solution will be

$$H\ (m,\ z)=B\ (m)\ e^{-\sqrt{(m^2-k^2{}_2)}z}+C\ (m)\ e^{\sqrt{(m^2-k^2{}_2)}z}, \quad for\ z'>z>0. \tag{A-2}$$

and in the third subspace z0, a solution is

$$H\ (m,\ z)=D\ (m)\ e^{\sqrt{(m^2-k^2{}_2)}z}\ for\ zo. \tag{A-3}$$

Let us write the boundary conditions in terms of the continuity of the tangential components of both the magnetic and electric fields at the interface z = z′:

$$A\ (m)\ e^{-\sqrt{(m^2-k^2{}_1)}z'}=B\ (m)\ e^{-\sqrt{(m^2-k^2{}_2)}z'}+C\ (m)\ e^{\sqrt{(m^2-k^2{}_2)}z'} \tag{A-4}$$

and

$$\frac{\sigma_2}{\sigma_1}(-\sqrt{m^2-k_1^2})\ A\ (m)\ e^{-\sqrt{(m^2-k^2{}_1)}z'}=$$
$$(-\sqrt{m^2-k_2^2})\ \ B\ (m)\ e^{-\sqrt{m^2-k^2{}_2}z'}+(\ \sqrt{m^2-k_2^2})\ \ c\ (m)\ e^{\sqrt{(m^2-k^2{}_2)}z'}. \tag{A-5}$$

At the location z = 0, we impose as the first boundary condition ghe continuity of the magnetic field tangential component:

$$B\ (m)=C\ (m)=D\ (m). \tag{A-6}$$

The second boundary condition comprises the jump discontinuity of its first derivative. In order to find it let us integrate equation (17) with respect to z over a small segment $[-\tau,\ +\tau]$:

$$\frac{\partial}{\partial_z}H\ (m,\ \tau)-\frac{\partial}{\partial_z}H\ (m,\ -\tau)=\frac{I\Delta Z}{2\pi a}sin\ (ma). \tag{A-7}$$

This yields

$$(-\sqrt{m^2-k_2^2})\ B\ (m)+(\sqrt{m^2-k_2^2})\ C\ (m)-(\sqrt{m^2-k_2^2})\ D\ (m)=\frac{I\Delta z}{2\pi a}sin\ (ma). \tag{A-8}$$

We have four equations with four unknowns. Solving equations for unknown coefficients A, B, C and D, we obtain

$$A\ (m)=-\frac{I\Delta z\ sin\ (ma)\ e^{-\sqrt{(m^2-k^2{}_2)}\ z'+\sqrt{(m^2-k^2{}_1)}\ z'}}{2\pi a\ \ [\sqrt{m^2-k_2^2}+(\frac{\Theta_2}{\Theta_1})\sqrt{m^2-k_1^2}]}, \tag{A-9}$$

$$B\ (m)=-\frac{I\Delta z\ sin\ (ma)}{4\pi a\ \sqrt{m^2-k_2^2}}, \tag{A-10}$$

$$C\ (m)=-\frac{I\Delta z\ sin\ (ma)\ e^{-2\sqrt{m^2-k^2{}_2}z'}\ [\sqrt{m^2-k_2^2}-(\frac{\Theta_2}{\Theta_1})\ \sqrt{m^2-k_1^2}]}{4\pi a\ \sqrt{m^2-k_2^2}\qquad [\sqrt{m^2-k_2^2}+(\frac{\Theta_2}{\Theta_1})\sqrt{m^2-k_1^2}]}, \tag{A-11}$$

and

$$D(m) = -\frac{I\Delta z \sin(ma)}{4\pi a\sqrt{m^2-k_2^2}} - \tag{A-12}$$

$$-\frac{I\Delta z \sin(ma)\, e^{-2\sqrt{m^2-k_2^2}\,z'}}{4\pi a\sqrt{m^2-k_2^2}} \frac{[\sqrt{m^2-k_2^2}-(\frac{\Theta_2}{\Theta_1})\sqrt{m^2-k_1^2}]}{[\sqrt{m^2-k_2^2}+(\frac{\Theta_2}{\Theta_1})\sqrt{m^2-k_1^2}]}$$