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SPECTRAL REPRESENTATION OF A LOVE-TYPE OPERATOR

Abstract. The wave-operator corresponding to a horizontally polarized shear wave travelling in an elastic layer of uniform thickness lying between two half-spaces is studied. The Green's function for the problem is derived and is then used to get a spectral representation of this operator.

INTRODUCTION

The spectral representation of an operator arising from certain wave propagation problems is useful in dealing with transmission, reflection and diffraction of waves across a horizontal discontinuity. This approach is based upon two steps: first, we find the Green's function associated with the problem. Secondly, we integrate it around a large circle enclosing all the singularities. In the case of finite depth problems, the sum of residues at the poles gives the representation of the delta function as a series in eigenfunctions. However, if we deal with the wave propagation in media with infinite depth, the Green's function, in addition to poles, possesses branch point singularities. The resulting representation then involves the sum of residues at the poles and a branch cut integral over a portion of the real axis (Friedman, 1956; Stakgold, 1979).

Kazi (1976) has used this approach to obtain a spectral representation of the operator arising from horizontally polarized shear waves propagating in a soft layer of uniform thickness overlying a harder half-space. Such waves are of interest in earthquake engineering and seismology and are commonly known as Love waves. Kazi's results were subsequently applied to various problem involving Love waves by, among others, Kazi (1979), Niazy and Kazi (1980) and Madja et al. (1985).

We are interested here in a spectral representation of horizontally polarized shear waves travelling in a layer of uniform thickness between two half-spaces. Such a model could adequately describe a soft layer lying deep under the Earth. Due to the similar nature of the wave and its dispersion relation, we call it a Love-type wave and the operator, arising from the equation of motion, a Love-type operator (Fig. 1).

EQUATION OF MOTION

The elastic layer $0 \leq z \leq H$ with uniform thickness H is assumed to be sandwiched between two homogeneous half-spaces $z < 0$ and $z > H$. The rigidity, shear velocity and density of the respective medium are denoted by μ_i, β_i ($\beta_1 < \beta_2 < \beta_3$) and ϱ_i for $i = 1, 2, 3$. The subscripts 1, 2, and 3 refer to, respectively, the upper medium, the intermediate layer and the lower medium. The geometry of the problem is shown in Fig. 1. Let v(x,z,t) be the horizontal component of displacement; then the equation of motion for horizontally polarized shear wave is

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$$\frac{\partial}{\partial x}\left[\mu(z)\frac{\partial v(x,z,t)}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu(z)\frac{\partial v(x,z,t)}{\partial z}\right]=\varrho(z)\frac{\partial^2 v(x,z,t)}{\partial t^2}, \tag{1}$$

where

$$\begin{aligned}\mu(z) &= \mu_1, \quad z<0,\\ &= \mu_2, \quad 0\le z\le H,\\ &= \mu_3, \quad z>H,\end{aligned} \tag{2}$$

and

$$\begin{aligned}\varrho(z) &= \varrho_1, \quad z<0,\\ &= \varrho_2, \quad 0\le z\le H,\\ &= \varrho_3, \quad z>H,\end{aligned} \tag{3}$$

are piecewise constant functions and $v(x, z, t)=v_i(x, z, t)$ denotes the displacement in the respective medium for $i=1, 2, 3$. We assume the motion to be time harmonic and write

$$v(x,z,t)=V(z)\exp[i(\omega t-kx)], \tag{4}$$

where ω is the angular frequency and k is the wave number. Omitting the time factor $e^{i\omega t}$, we can write the equation of motion as

$$L(V)=\frac{d}{dz}\left(\mu\frac{dV}{dz}\right)+(\omega^2\varrho-k^2\mu)V=0, \tag{5}$$

where

$$V(z)=V_i(z), \quad i=1,2,3.$$

$V_i(z)$ satisfy the following equations:

$$\frac{d^2V_1}{dz^2}-\sigma_1^2V_1=0,\quad \sigma_1^2=(\lambda-\frac{\omega^2}{\beta_1^2}),\quad \lambda=k^2,\quad \beta_1^2=\frac{\mu_1}{\varrho_1},\quad z<0. \tag{6}$$

$$\frac{d^2V_2}{dz^2}+\sigma_2^2V_2=0,\quad \sigma_2^2=(\frac{\omega^2}{\beta_2^2}-\lambda),\quad \beta_2^2=\frac{\mu_2}{\varrho_2},\quad 0\le z\le H. \tag{7}$$

$$\frac{d^2V_3}{dz^2}-\sigma_3^2V_3=0,\quad \sigma_3^2=(\lambda-\frac{\omega^2}{\beta_3^2}),\quad \beta_3^2=\frac{\mu_3}{\varrho_3},\quad z>H. \tag{8}$$

In order to satisfy the radiation condition (cf. eqn. (17)), the appropriate choice of the branch (Riemann sheet) of σ_j $(j=1, 2, 3)$ is given by

$$\sigma_j=\left((\lambda-\frac{\omega^2}{\beta_j^2})\right)^{1/2}\quad (j=1,2,3).$$

Troughout we will restrict our selves to this Riemann sheet. The interface conditions are

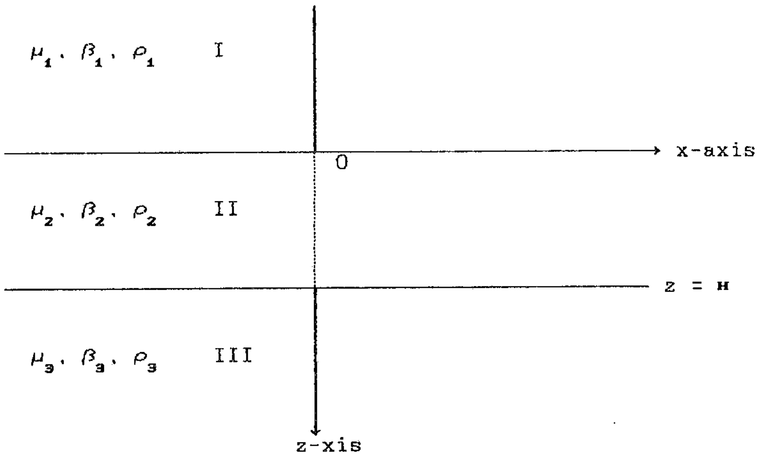

Fig. 1 — Geometry of the problem.

$$V_1\ (0)=V_2\ (0), \tag{9a}$$

$$\mu_1\ V'_1\ (0)=\mu_2\ V'_2\ (0), \tag{9b}$$

and

$$V_2\ (H)=V_3\ (H), \tag{10a}$$

$$\mu_2\ V'_2\ (H)=\mu_3\ V'_3\ (H). \tag{10b}$$

Here prime denotes differentiation with respect to z. The conditions at $\pm\ \infty$ are

$$\int_{-\infty}^{0} \mu\ |\ V\ (z)\ |^2\ dz<\infty, \tag{11}$$

and

$$\int_{0}^{\infty} \mu\ |\ V\ (z)\ |^2\ dz<\infty. \tag{12}$$

We thus have a self-adjoint Sturm-Liouville system that is singular in the half-spaces $z<0$ and $z>H$ and is regular in the layer $0\leq z\leq H$. As noted by Kazi (1976), the problem in the half-spaces is in the limit point case at infinity, and so the requirement that the solution must be of finite μ-norm is adequate for defining the solution precisely.

GREEN'S FUNCTION

Let $G\ (z,\ \xi;\ \lambda)=G_{ij}$, i, j=1, 2, 3 be the Green's function associated with the problem. Here the first subscript refers to the z-interval and the second subscript refers to the ξ-interval. Thus G_{13} , for example, would refer to the Green's function $G\ (z,\ \xi;\ \lambda)$ for z in the half-space $z<0$ and ξ lying in the half-space $z>H$.

G_{ij} satisfy:

(G_1) *$G_{ij}\ (z,\ \xi;\ \lambda)$ is a continuous function of z;*

(G_2) G_{ij} ($i\neq j$) possesses a continuous first order derivative at each point z of the ith medium;

G_{ij} (i=j) possesses a continuous first order derivative at every point z of the ith medium except z=ξ and it has a jump discontinuity given by:

$$G'_{ii}(\xi^{+}, \xi; \lambda) - G'_{ii}(\xi^{-}, \xi; \lambda) = 1/\mu_i(\xi);$$

(G_3) *If* $i \neq j$, $L(G_{ij}) = 0.$

If $i = j$, $L(G_{ij}) = \delta(z-\xi);$

(G_4) G (z, ξ; λ) satisfy the interface conditions (9a) to (10b). We determine G_{ij} as follows:

(1) For j=1, ξ is in the half-space z<0 and G_{11}, G_{21}, G_{31} satisfy the differential equations

$$\frac{d^2 G_{11}}{dz^2} - \sigma_1^2 G_{11} = \delta(z-\xi), \tag{13}$$

$$\frac{d^2 G_{21}}{dz^2} + \sigma_2^2 G_{21} = 0, \tag{14}$$

and

$$\frac{d^2 G_{31}}{dz^2} - \sigma_3^2 G_{31} = 0, \tag{15}$$

together with the following conditions:

$$\int_{-\infty}^{0} \mu(z) |G_{11}(z)|^2 dz < \infty \tag{16a}$$

$$G_{11} = G_{21}, \qquad at\ z=0, \tag{16b}$$

$$\mu_1 G'_{11} = \mu_2 G'_{21}, \qquad at\ z=0, \tag{16c}$$

$$G_{21} = G_{31}, \qquad at\ z=H, \tag{16d}$$

$$\mu_2 G'_{21} = \mu_3 G'_{31}, \qquad at\ z=H, \tag{16e}$$

$$G_{11}(\xi^{+}, \xi; \lambda) = G_{11}(\xi^{-}, \xi; \lambda), \tag{16f}$$

$$G'_{11}(\xi^{+}, \xi; \lambda) - G'_{11}(\xi^{-}, \xi; \lambda) = 1/\mu_1, \tag{16g}$$

and

$$\int_{0}^{\infty} \mu(z) |G_{31}(z)|^2 dz < \infty. \tag{16h}$$

We find

$$G_{11}(z, \xi; \lambda) = \frac{M}{2\mu_1 \sigma_1 \Delta} e^{\sigma_1 (z+\xi)} - \frac{1}{2\mu_1 \sigma_1} [e^{\sigma_1 (z-\xi)} H(\xi - z) + e^{-\sigma_1 (z-\xi)} H(z-\xi)], \tag{17}$$

where

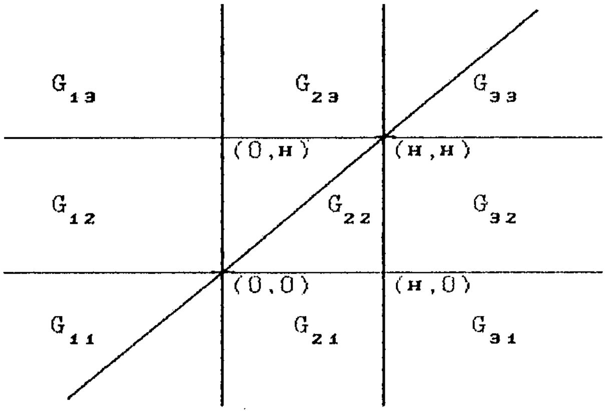

Fig. 2 — The character of the Green's function.

$$M=\mu_2\sigma_2\,(\mu_2\sigma_2\,\tan\sigma_2H-\mu_3\sigma_3)+\mu_1\sigma_1\,(\mu_3\sigma_3\,\tan\sigma_2H+\mu_2\sigma_2), \tag{18}$$

$$\Delta=\mu_2\sigma_2\,(\mu_2\sigma_2\,\tan\sigma_2H-\mu_3\sigma_3)-\mu_1\sigma_1\,(\mu_3\sigma_3\,\tan\sigma_2H+\mu_2\sigma_2), \tag{19}$$

and H(x) is the Heaviside function.

$$G_{21}\,(z,\,\xi;\,\lambda)=\frac{e^{\,\sigma_1\xi}\,P(z)}{\Delta\cos\sigma_2H}, \tag{20}$$

where

$$P(z)=\mu_2\sigma_2\,\cos\sigma_2\,(H-z)+\mu_3\sigma_3\,\sin\sigma_2\,(H-z), \tag{21}$$

$$G_{31}\,(z,\,\xi;\,\lambda)=\frac{\mu_2\sigma_2\,e^{\,\sigma_1\xi}\,e^{-\sigma_3\,(z-H)}}{\Delta\cos\sigma_2H}. \tag{22}$$

(2) If ξ is in the layer $0\leqslant z\leqslant H$, G_{12}, G_{22} and G_{32} satisfy the differential equations

$$\frac{d^2G_{12}}{dz^2}-\sigma_1^2\,G_{12}=0, \tag{23}$$

$$\frac{d^2G_{22}}{dz^2}+\sigma_2^2\,G_{22}=\delta\,(z-\xi), \tag{24}$$

$$\frac{d^2G_{32}}{dz^2}-\sigma_3^2\,G_{32}=0, \tag{25}$$

together with the conditions

$$\int_{-\infty}^{0}\mu(z)|G_{12}(z)|^2dz<\infty\ , \tag{26a}$$

$$G_{12}=G_{22}, \qquad \text{at } z=0, \tag{26b}$$

$$\mu_1G'_{12}=\mu_2\,G'_{22}, \qquad \text{at } z=0, \tag{26c}$$

$$G_{22}=G_{32}, \qquad \text{at } z=H, \tag{26d}$$

$$\mu_2 G'_{22}=\mu_3\, G'_{32}, \qquad \text{at } z=H, \tag{26e}$$

$$\int_0^{\infty} \mu(z)|G_{32}(z)|^2 dz < \infty, \tag{26f}$$

$$G_{22}\,(\xi^+,\ \xi;\ \lambda)=G_{22}\,(\xi^-,\ \xi;\ \lambda), \tag{26g}$$

and

$$G'_{22}\,(\xi^+,\ \xi;\ \lambda)-G'_{22}\,(\xi^-,\ \xi;\ \lambda)=1/\mu_2, \tag{26h}$$

we find that

$$G_{12}\,(z,\ \xi;\ \lambda)=\frac{B\ e^{\sigma_1 z}}{\Delta \cos\sigma_2 H}, \tag{27}$$

where

$$B=\mu_2\sigma_2\cos\sigma_2\,(H-\xi)+\mu_3\sigma_3\,\sin\sigma_2\,(H-\xi). \tag{28}$$

$$G_{22}(z,\xi;\ \lambda)=\frac{K\mu_1\sigma_1\sin\sigma_2\xi\sin\sigma_2 z+N\mu_2\sigma_2\cos\sigma_2\xi\cos\sigma_2 z}{\mu_2\sigma_2\Delta\cos\sigma_2 H}+$$
$$\frac{1}{\mu_2\sigma_2\Delta\cos\sigma_2 H}[\{K\mu_2\sigma_2\sin\sigma_2\xi\cos\sigma_2 z+N\mu_1\sigma_1\cos\sigma_2\xi\sin\sigma_2 z\}H(\xi-z)+$$
$$\{K\mu_2\sigma_2\sin\sigma_2 z\cos\sigma_2\xi+N\mu_1\sigma_1\sin\sigma_2\xi\cos\sigma_2 z\}H(z-\xi)], \tag{29}$$

with

$$K=\mu_2\sigma_2\sin\sigma_2 H-\mu_3\sigma_3\cos\sigma_2 H, \quad N=\mu_2\sigma_2\cos\sigma_2 H+\mu_3\sigma_3\sin\sigma_2 H,$$

and

$$G_{32}\,(z,\ \xi;\ \lambda)=\frac{(\mu_2\,\sigma_2\,\cos\sigma_2\,\xi+\mu_1\,\sigma_1\,\sin\sigma_2\,\xi)\ e^{-\sigma_3\,(z-H)}}{\Delta\,\cos\sigma_2 H}. \tag{30}$$

(3) If ξ lies in the lower half-space $z>H$, then G_{13}, G_{23}, G_{33} satisfy the differential equations

$$\frac{d^2\,G_{13}}{dz^2}-\sigma_1^2\,G_{13}=0, \tag{31}$$

$$\frac{d^2\,G_{23}}{dz^2}+\sigma_2^2\,G_{23}=0, \tag{32}$$

$$\frac{d^2 G_{33}}{dz^2}-\sigma_3^2 G_{33}=\delta\,(z-\xi), \tag{33}$$

together with the conditions

$$\int_{-\infty}^{0} \mu(z) |G_{13}(z)|^2 dz < \infty, \qquad (34a)$$

$$G_{13} = G_{23}, \qquad \text{at } z=0, \qquad (34b)$$

$$\mu_1 G'_{13} = \mu_2 G'_{23}, \qquad \text{at } z=0, \qquad (34c)$$

$$G_{23} = G_{33}, \qquad \text{at } z=H, \qquad (34d)$$

$$\mu_2 G'_{23} = \mu_3 G'_{33}, \qquad \text{at } z=H, \qquad (34e)$$

$$\int_{0}^{\infty} \mu(z) \mid G_{33}(z) \mid^2 dz < \infty, \qquad (34f)$$

$$G_{33}(\xi^+, \xi; \lambda) = G_{33}(\xi^-, \xi; \lambda), \qquad (34g)$$

and the jump condition

$$G'_{33}(\xi^+, \xi; \lambda) - G'_{33}(\xi^-, \xi; \lambda) = 1/\mu_3, \qquad (34h)$$

It follows that

$$G_{13}(z, \xi; \lambda) = \frac{\mu_2 \sigma_2 e^{-\sigma_3(\xi-H)} e^{\sigma_1 z}}{\Delta \cos\sigma_2 H}, \qquad (35)$$

$$G_{23}(z, \xi; \lambda) = \frac{(\mu_2 \sigma_2 \cos\sigma_2 z + \mu_1 \sigma_1 \sin\sigma_2 z) e^{-\sigma_3(\xi-H)}}{\Delta \cos\sigma_2 H}, \qquad (36)$$

and

$$G_{33}(z, \xi; \lambda) = \frac{C}{2\mu_3 \sigma_3 \Delta} e^{-\sigma_3(z+\xi-2H)} - \frac{1}{2\mu_3 \sigma_3} \left[e^{\sigma_3(z-\xi)} H(\xi - z) + e^{-\sigma_3(z-\xi)} H(z-\xi) \right], \qquad (37)$$

where

$$C = \mu_2 \sigma_2 (\mu_2 \sigma_2 \tan\sigma_2 H + \mu_3 \sigma_3) + \mu_1 \sigma_1 (\mu_3 \sigma_3 \tan\sigma_2 H - \mu_2 \sigma_2). \qquad (38)$$

We have thus determined $G_{ij}(z, \xi; \lambda)$ and therefore $G(z, \xi; \lambda)$. We note that $G_{ij} = G_{ji}$ so that the Green's function is symmetric as expected.

SPECTRAL REPRESENTATION

The essential step in obtaining the spectral representation is to integrate the Green's function $G(z, \xi, \lambda)$ obtained in the previous section around a large circle $|\lambda| = R$ in the complex λ—plane. The Green's function, in addition to simple poles, possesses branch point singularities. The spectrum is the disjoint union of the discrete and the continuous spectrum, giving rise to proper and improper eigenfunctions. The continuous spectrum will be the set of points on the branch-cut along a portion of the real axis, and the discrete spectrum will be the set of

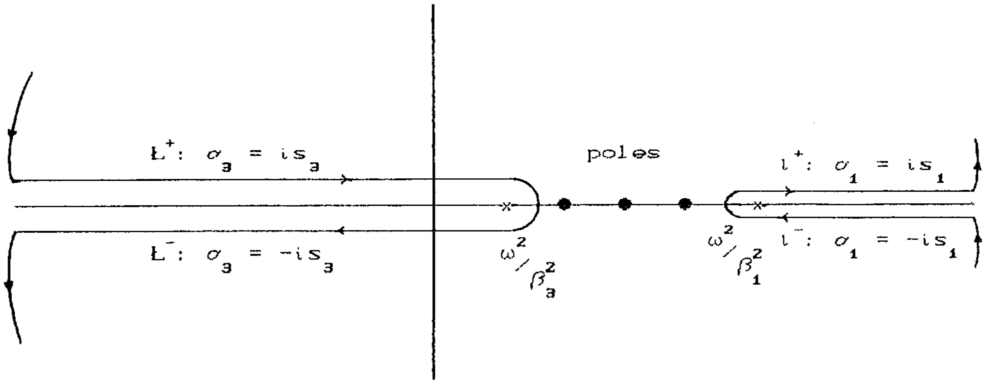

Fig. 3 — The contour of integration in the complex λ-plane.

poles lying on the real axis. The sum of residues at the poles and the contribution from the branch-cuts yield the following representation of the delta function in terms of proper eigenfunctions $\{\phi_n(z)\}$ and improper eigenfunctions $\{\psi(z, \lambda)\}$ and $\{\chi(z, \lambda)\}$ (see Kazi, 1976):

$$\lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G(z, \xi; \lambda)\, d\lambda = \sum_n \phi^n(z)\, \overline{\phi^n(\xi)} + \int \psi(z, \lambda)\, \overline{\psi(\xi, \lambda)}\, d\lambda +$$

$$\int \chi(z, \lambda)\, \overline{\chi(\xi, \lambda)}\, d\lambda = \frac{\delta(z-\xi)}{\mu(\xi)} \quad . \tag{39}$$

We shall use this formula, step by step, for each $G_{ij}(z, \xi; \lambda)$:

(i) first, we consider

$$I_{11} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{11}(z, \xi; \lambda)\, d\lambda = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{1}{2\mu_1 \sigma_1} \left\{ \frac{M e^{\sigma_1 (z+\xi)}}{\Delta} - e^{-\sigma_1 |z-\xi|} \right\} d\lambda, \tag{40}$$

where M and Δ are given by eqns. (18) and (19), respectively. We notice that $\sigma_1=0$ gives rise to the branch point $\lambda = \frac{\omega^2}{\beta_1^2}$, $\sigma_3=0$ gives the branch point $\lambda = \frac{\omega^2}{\beta_3^2}$ and the poles of the integrand are roots of the equation (Fig. 3)

$$\Delta = \mu_2 \sigma_2 (\mu_2 \sigma_2 \tan\sigma_2 H - \mu_3 \sigma_3) - \mu_1 \sigma_1 (\mu_3 \sigma_3 \tan\sigma_2 H + \mu_2 \sigma_2) = 0,$$

which is the dispersion relation for Love-type waves propagating in a layer of uniform thickness sandwiched between two half-spaces. The poles are all simple, finite in number, and are located in the open interval $\left(\frac{\omega^2}{\beta_3^2}, \frac{\omega^2}{\beta_1^2} \right)$. The set of these poles constitutes the discrete spectrum. The continuous spectrum arises from the integral over the branch cuts at $\frac{\omega^2}{\beta_3^2}$ and $\frac{\omega^2}{\beta_1^2}$. The sum of the residues at the poles $\{\lambda_n\}$ is given by

$$-\sum_{n=1}^{N} \frac{(M)_{\lambda=\lambda_n} \exp\{\sigma_1^{(n)}(z+\xi)\}}{2\mu_1 \sigma_1^{(n)} \frac{\partial}{\partial\lambda}[\Delta]_{\lambda=\lambda_n}} = \sum_{n=1}^{N} \phi_1^{(n)}(z)\,\phi_1^{(n)}(\xi), \tag{41}$$

where

$$\sigma_1^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_1^2}\right)^{1/2}, \quad \sigma_2^{(n)} = \left(\frac{\omega^2}{\beta_2^2} - \lambda_n\right)^{1/2}, \quad \sigma_3^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_3^2}\right)^{1/2},$$

and

$$\phi_1^{(n)}(z) = \left[\frac{1}{2\mu_1 \sigma_1^{(n)}} \left\{\frac{M}{\frac{\partial}{\partial\lambda}(-\Delta)}\right\}_{\lambda=\lambda_n}\right]^{1/2} \exp(\sigma_1^{(n)} z). \tag{42}$$

The contribution from the branch cuts for the two branch points $\frac{\omega^2}{\beta_1^2}$ (or $\frac{\omega^2}{\beta_3^2}$) can be evaluated by taking Re $(\sigma_1)>0$ {Re $(\sigma_3)>0$} on l^+, putting $\sigma_1 = is_1$ on l^+, and $\sigma_1 = -is_1$ on l^- (or $\sigma_3 = is_3$ on L^+ and $\sigma_3 = -is_3$ on L^-), where $s_1 = \left(\frac{\omega^2}{\beta_1^2} - \lambda\right)^{1/2}$ is real and positive for $\lambda < \frac{\omega^2}{\beta_1^2}$ [or $s_3 = \left(\frac{\omega^2}{\beta_3^2} - \lambda\right)^{1/2}$ is real and positive for $\lambda < \frac{\omega^2}{\beta_3^2}$]. These contributions, respectively, can be shown to be

$$I'_{11} = \int (G_{11}^{+} - G_{11}^{-})\, d\lambda = -\frac{1}{\pi} \int_{\omega^2/\beta_1^2}^{\infty} \frac{1}{\mu_1 s_1} \sin(\theta + s_1 z) \sin(\theta + s_1 \xi)\, d\lambda,$$

$$= -\int_{\omega^2/\beta_1^2}^{\infty} \psi_1(z, \lambda)\, \psi_1(\xi, \lambda)\, d\lambda \tag{43}$$

where

$$\theta = \tan^{-1}\left\{\frac{\mu_1 s_1 (\mu_3 \sigma_3 \tan\sigma_2 H + \mu_2 \sigma_2)}{\mu_2 \sigma_2 (\mu_2 \sigma_2 \tan\sigma_2 H - \mu_3 \sigma_3)}\right\}.$$

$$\psi_1(z, \lambda) = G_\lambda \sin(\theta + s_1 z), \qquad G_\lambda = \frac{1}{\sqrt{\pi \mu_1 s_1}},$$

and

$$I''_{11}=-\frac{1}{\pi}\int_{-\infty}^{\omega^2/\beta_3^2}\frac{\mu_2^2\,\sigma_2^2\,\sigma_3\,s_3\,\exp\,\{\sigma_1\,(z+\xi)\}}{(p^2+q^2)\,\cos^2\,\sigma_2\,H}\,d\lambda,$$

$$=-\int_{-\infty}^{\omega^2/\beta_3^2}\chi_1\,(z,\,\lambda)\,\chi_1\,(\xi,\,\lambda)\,d\lambda, \tag{44}$$

where

$$p=\cos\varphi=\mu_2\,\sigma_2\,(\mu_2\,\sigma_2\,\tan\,\sigma_2\,H-\mu_1\,\sigma_1),\;\; q=\sin\varphi=\mu_3\,s_3\,(\mu_1\,\sigma_1\,\tan\,\sigma_2\,H+\mu_2\,\sigma_2)\,.$$

$$\chi_1\,(z,\,\lambda)=\frac{\mu_2\,\sigma_2\,\mu_3\,s_3\,e^{\sigma_1 z}}{\sqrt{\pi\mu_3\,s_3}\,\cos\sigma_2\,H}\quad .$$

Therefore

$$I_{11}=\sum_{n=1}^{N}\phi_1^{(n)}\,(z)\,\phi_1^{(n)}\,(\xi)-\int_{-\infty}^{\omega^2/\beta_3^2}\chi_1\,(z,\,\lambda)\,\chi_1\,(\xi,\,\lambda)\,d\lambda-\int_{\omega^2/\beta_1^2}^{\infty}\psi_1\,(z,\,\lambda)\,\psi_1\,(\xi,\,\lambda)\,d\lambda. \tag{45}$$

(ii) Next, we consider

$$I_{21}=\lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R}G_{21}\,(z,\,\xi;\,\lambda)\,d\lambda=\lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R}\frac{e^{\sigma_1\xi}\,P\,(z)}{\Delta\cos\sigma_2\,H}\,d\lambda\,, \tag{46}$$

where P (z) is given by (21).

The branch line contribution at $\lambda=\frac{\omega^2}{\beta_1^2}$ (or $\lambda=\frac{\omega^2}{\beta_3^2}$) is given by putting $\sigma_1=is_1$ on l^+, and $\sigma_1=-is_1$ on l^- (or $\sigma_3=is_3$ on L^+ and $\sigma_3=-is_3$ on L^-), so that for $\lambda=\frac{\omega^2}{\beta_1^2}$,

$$G_{21}^+-G_{21}^-=\frac{2i\,(m\sin s_1\,\xi+n\cos s_1\,\xi)\,P\,(z)}{m^2+n^2}\quad ,$$

where

$$m=\mu_2\,\sigma_2\,(\mu_2\,\sigma_2\,\tan\sigma_2\,H-\mu_3\,\sigma_3),\quad n=\mu_1\,s_1\,(\mu_3\,\sigma_3\,\tan\sigma_2\,H+\mu_2\,\sigma_2).$$

Hence

$$I'_{21}=-\frac{1}{\pi}\int_{\omega^2/\beta_1^2}^{\infty}\frac{\sin\,(\theta+s_1\,\xi)\,P\,(z)}{\cos\sigma_2\,H}\,d\lambda,$$

$$I'_{21} = -\int_{\omega^2/\beta_1^2}^{\infty} \frac{\sin(\theta + s_1 \xi)}{\sqrt{\pi\mu_1 s_1}} \frac{\mu_1 s_1 P(z)}{\sqrt{\pi\mu_1 s_1} \cos\sigma_2 H} d\lambda,$$

$$= -\int_{\omega^2/\beta_1^2}^{\infty} \psi_1(\xi, \lambda)\, \psi_2(z, \lambda)\, d\lambda, \tag{47}$$

with

$$\psi_2(z, \lambda) = G_\lambda \frac{\mu_1 s_1 P(z)}{\cos\sigma_2 \mathrm{H}} .$$

For the branch line contribution at $\lambda = \frac{\omega^2}{\beta_3^2}$, we get

$$I''_{21} = -\int_{-\infty}^{\omega^2/\beta_3^2} \left\{ \frac{\mu_2 \sigma_2 \mu_3 s_3}{\sqrt{\pi\mu_3 s_3}} \frac{e^{\sigma_1 \xi}}{\cos\sigma_2 H} \right\} \left\{ \frac{\mu_3 s_3}{\sqrt{\pi\mu_3 s_3}} \frac{Q(z)}{\cos\sigma_2 H} \right\} d\lambda,$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2} \chi_1(\xi, \lambda)\, \chi_2(z, \lambda)\, d\lambda, \tag{48}$$

where

$$\chi_2(z, \lambda) = \frac{\mu_3 s_3 Q(z)}{\sqrt{\pi\mu_3 s_3} \cos\sigma_2 H}, \quad Q(z) = \mu_2 \sigma_2 \cos\sigma_2 z + \mu_1 \sigma_1 \sin\sigma_2 z.$$

The contribution from poles is given by

$$-\sum_{n=1}^{N} \frac{\exp(\sigma_1^{(n)} \xi)\, [P(z)]_{\lambda=\lambda_n}}{\cos\sigma_2^{(n)} H \frac{\partial}{\partial\lambda} [\Delta]_{\lambda=\lambda_n}} .$$

$$= \left[\frac{1}{2\mu_1 \sigma_1^{(n)}} \left\{ \frac{M}{\frac{\partial}{\partial\lambda}(-\Delta)} \right\}_{\lambda=\lambda_n} \right]^{1/2} e^{\sigma_1^{(n)} \xi} \left[2\mu_1 \sigma_1^{(n)} \left\{ \frac{1}{M \frac{\partial}{\partial\lambda}(-\Delta)} \right\}_{\lambda=\lambda_n} \right]^{1/2} \frac{[P(z)]_{\lambda=\lambda_n}}{\cos\sigma_2^{(n)} H},$$

$$= \sum_{n=1}^{N} \phi_1^{(n)}(\xi)\, \phi_2^{(n)}(z), \tag{49}$$

where

$$\phi_2^{(n)}(z) = \left[2\mu_1 \sigma_1^{(n)} \left\{ \frac{1}{M \frac{\partial}{\partial\lambda}(-\Delta)} \right\}_{\lambda=\lambda_n} \right]^{1/2} \frac{[P(z)]_{\lambda=\lambda_n}}{\cos \sigma_2^{(n)} H} .$$

Hence

$$I_{21} = \sum_{n=1}^{N} \phi_2^{(n)}(z)\, \phi_1^{(n)}(\xi) - \int_{-\infty}^{\omega^2/\beta_3^2} \chi_2(z, \lambda)\, \chi_1(\xi, \lambda)\, d\lambda - \int_{\omega^2/\beta_1^2}^{\infty} \psi_2(z, \lambda)\, \psi_1(\xi, \lambda)\, d\lambda. \tag{50}$$

(iii) In a way similar to above.

$$I_{31} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{31}(z, \xi; \lambda)\, d\lambda = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{\mu_2 \sigma_2 e^{\sigma_1 \xi} e^{-\sigma_3 (z-H)}}{\Delta \cos\sigma_2 H}\, d\lambda. \tag{51}$$

The poles are zeros of Δ as above while the branch points are also the same. We can calculate the residues at poles and the branch points contribution to write

$$I_{31} = \sum_{n=1}^{N} \phi_3^{(n)}(z)\, \phi_1^{(n)}(\xi) - \int_{-\infty}^{\omega^2/\beta_3^2} \chi_3(z, \lambda)\, \chi_1(\xi, \lambda)\, d\lambda - \int_{\omega^2/\beta_1^2}^{\infty} \psi_3(z, \lambda)\, \psi_1(\xi, \lambda)\, d\lambda. \tag{52}$$

where

$$\phi_3(z) = \left[2\mu_1 \sigma_1^{(n)} \left\{ \frac{1}{M \frac{\partial}{\partial\lambda}(-\Delta)} \right\}_{\lambda=\lambda_n} \right]^{1/2} \frac{\mu_2 \sigma_2^{(n)} e^{-\sigma_3^{(n)} (z-H)}}{\cos \sigma_2^{(n)} H}, \quad \varphi = \tan^{-1} ({}^q/p),$$

$$\chi_3(z, \lambda) = \frac{\sin\{\varphi - s_3 (z-H)\}}{\sqrt{\pi \mu_3}\, s_3}, \quad \psi_3(z, \lambda) = G_\lambda \frac{\mu_2 \sigma_2 \mu_1 s_1 e^{-\sigma_3 (z-H)}}{\cos\sigma_2 H} .$$

All the other integrals can be manipulated in the same manner as (i) to (iii). Thus we can write

$$I_{ij} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{ij}(z, \xi; \lambda)\, d\lambda = \sum_{n=1}^{N} \phi_i^{(n)}(z)\, \phi_j^{(n)}(\xi) -$$

$$\int_{-\infty}^{\omega^2/\beta_3^2} \chi_i (z, \lambda)\, \chi_j (\xi, \lambda)\, d\lambda - \int_{\omega^2/\beta_1^2}^{\infty} \psi_i (z, \lambda)\, \psi_j (\xi, \lambda)\, d\lambda, \tag{53}$$

(i, j = 1, 2, 3) with G_{ij} given by eqns. (27), (29), (30), (35), (36), and (37).

From eqns. (39) and (53), we obtain the following representation of delta functions:

$$\delta (z-\xi) = \sum_{n=1}^{N} \mu (\xi)\, \phi^{(n)} (z)\, \phi^{(n)} (\xi) - \int_{-\infty}^{\omega^2/\beta_3^2} \mu (\xi)\, \chi (z, \lambda)\, \chi (\xi, \lambda)\, d\lambda - \int_{\omega^2/\beta_1^2}^{\infty} \mu (\xi)\, \psi (z, \lambda)\, \psi (\xi, \lambda)\, d\lambda, \tag{54}$$

where $\mu (\xi)$ is given by eqn. (2) and

$$\begin{aligned} \phi^{(n)} (z) &= \phi_1^{(n)} (z), && z<0, \\ &= \phi_2^{(n)} (z), && 0\leq z\leq H, \\ &= \phi_3^{(n)} (z), && z>H, \end{aligned} \tag{55}$$

are the normalized eigenfunctions. The improper eigenfunctions are

$$\begin{aligned} \chi (z, \lambda) &= \chi_1(z, \lambda), && z<0, \\ &= \chi_2(z, \lambda), && 0\leq z\leq H, \\ &= \chi_3(z, \lambda), && z>H, \end{aligned} \tag{56}$$

and

$$\begin{aligned} \psi (z, \lambda) &= \psi_1(z, \lambda), && z<0, \\ &= \psi_2(z, \lambda), && 0\leq z\leq H, \\ &= \psi_3(z, \lambda), && z>H. \end{aligned} \tag{57}$$

If f (z) is of finite μ-norm over the integral $(-\infty, \infty)$, then the representation of f(z) in terms of eigenfunctions $\{\phi^{(n)} (z)\}$ and improper eigenfunctions $\chi (z, \lambda)$ and $\psi (z, \lambda)$ can be obtained by multiplying eqn. (54) by f (ξ) and integrating with respect to ξ from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty} f (\xi)\, \delta (z-\xi)\, d\xi = \sum_{n=1}^{N} \phi^{(n)} (z) \int_{-\infty}^{\infty} \mu (\xi)\, f (\xi)\, \phi^{(n)} (\xi)\, d\xi - \int_{-\infty}^{\omega^2/\beta_3^2} \chi (z, \lambda)\, d\lambda$$

$$\times \int_{-\infty}^{\infty} \mu(\xi) f(\xi) \chi(\xi, \lambda) d\xi - \int_{\omega^2/\beta_1^2}^{\infty} \psi(z, \lambda) d\lambda \int_{-\infty}^{\infty} \mu(\xi) f(\xi) \psi(\xi, \lambda) d\xi . \quad (58)$$

If we write

$$f_n = < f, \phi^{(n)} > = \int_{-\infty}^{\infty} \mu(\xi) f(\xi) \phi^{(n)}(\xi) d\xi , \quad (59)$$

$$f_\alpha = < f, \chi(\xi, \lambda) > = \int_{-\infty}^{\infty} \mu(\xi) f(\xi) \chi(\xi, \lambda) d\xi , \quad (60)$$

$$f_\beta = < f, \psi(\xi, \lambda) > = \int_{-\infty}^{\infty} \mu(\xi) f(\xi) \psi(\xi, \lambda) d\xi , \quad (61)$$

then eqn. (58) becomes

$$f(z) = \sum_{n=1}^{N} f_n \phi^{(n)}(z) - \int_{-\infty}^{\omega^2/\beta_3^2} f_\alpha \chi(z, \lambda) dz - \int_{\omega^2/\beta_1^2}^{\infty} f_\beta \psi(z, \lambda) dz. \quad (62)$$

In particular, we can write the following orthonormality relations:

$$\int_{-\infty}^{\infty} \mu(z) \phi^{(m)}(z) \phi^{(n)}(z) dz = < \phi^{(m)}, \phi^{(n)} > = \delta_{mn} , 1 \leq m, n \leq N. \quad (63)$$

$$\int_{-\infty}^{\infty} \mu(z) \chi(z, \lambda) \chi(z, \lambda') dz = < \chi(z, \lambda), \chi(z, \lambda') > = \delta(\lambda - \lambda'),$$
$$-\infty < \lambda, \lambda' < \omega^2/\beta_3^2 . \quad (64)$$

$$\int_{-\infty}^{\infty} \mu(z) \psi(z, \lambda) \psi(z, \lambda') dz = < \psi(z, \lambda), \psi(z, \lambda') > = \delta(\lambda - \lambda'),$$
$$\omega^2/\beta_1^2 \leq \lambda, \lambda' < \infty. \quad (65)$$

$$\int_{-\infty}^{\infty} \mu(z) \phi^m(z) \chi(z, \lambda) dz = 0 = < \phi^m, \chi > , 1 \leq m \leq N,$$
$$-\infty < \lambda, \lambda' < \omega^2/\beta_3^2 . \quad (66)$$

$$\int_{-\infty}^{\infty} \mu(z) \phi^m(z) \psi(z, \lambda) dz = 0 = < \phi^m, \psi >, 1 \leq m \leq N,$$
$$\omega^2/\beta_1^2 \leq \lambda, \lambda' < \infty. \quad (67)$$

$$\int_{-\infty}^{\infty} \mu(z) \chi(z, \lambda) \psi(z, \lambda) dz = 0 = < \chi, \psi >, -\infty < \lambda < \omega^2/\beta_3^2 .$$
$$\omega^2/\beta_1^2 < \lambda < \infty . \quad (68)$$

CONCLUSION

We have obtained the spectral representation of the two-dimensional Love-type operator associated with monochromatic SH-waves in a structure comprising a homogeneous layer between two homogeneous half-spaces. The spectral representation makes possible the solution of classes of problems associated with the transmission and reflection of Love-type waves at a horizontally discontinuous change either in elevation or in material properties of an embedded layer model.

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